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Three-point functions for M^N/S^N orbifolds with $\mathcal{N}=4$ supersymmetry

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Abstract

The D1-D5 system is believed to have an 'orbifold point' in its moduli space where its low energy theory is a $\mathcal{N}=4$ supersymmetric sigma model with target space M^N/S^N, where M is T^4 or $K3$. We study correlation functions of chiral operators in CFTs arising from such a theory. We construct a basic class of chiral operators from twist fields of the symmetric group and the generators of the super-conformal algebra. We find explicitly the 3-point functions for these chiral fields at large N; these expressions are 'universal' in that they are independent of the choice of M. We observe that the result is a significantly simpler expression than the corresponding expression for the bosonic theory based on the same orbifold target space.

1 Introduction

The D1-D5 system has been of great interest in recent developments in string theory. The system is described by a collection of n_5 D5 branes which are wrapped on a 4-manifold M (which can be a T^4 or a K^3) and n_1 D1 branes parallel to the noncompact direction of the D5 branes and bound to them. This system has been very important for issues related to black holes, since it yields, upon addition of momentum excitations, a supersymmetric configuration which has a classical (i.e. not Planck size) horizon. In particular, the Bekenstein entropy computed from the classical horizon area agrees with the count of microstates for the extremal and near extremal black holes [1]. Further, the low energy Hawking radiation from the hole can be understood in terms of a unitary microscopic process, not only qualitatively but also quantitatively, since one finds an agreement of spin dependence and radiation rates between the semiclassically computed radiation and the microscopic calculation [2]. The AdS/CFT correspondence conjecture gives a duality between string theory on a spacetime and a certain conformal field theory (CFT) on the boundary of this spacetime [3]. The D1-D5 system gives a CFT which is dual to the spacetime $AdS_3 \times S^3 \times M$.

While it is possible to use simple models for the low energy dynamics of the D1-D5 system when one is computing the coupling to massless modes of the supergravity theory (as was done for example in the computations of Hawking radiation from the D1-D5 microstate), it is believed that the exact description of this CFT must be in terms of a sigma model with target space being a deformation of the orbifold M^N/S^N, which is the symmetric orbifold of N copies of M. (Here $N = n_1 n_5$, and we must take the low energy limit of the sigma model to obtain the desired CFT.) In particular we may consider the 'orbifold point' where the target space is exactly the orbifold M^N/S^N with no deformation. It was suggested in [4][5] that this CFT does correspond to a certain point in the moduli space of the D1-D5 system.

Thus this orbifold theory would be dual to string theory on $AdS_3 \times S^3 \times M^4$, but at this orbifold point the string theory is expected to be in a strongly coupled domain where it cannot be approximated by tree level supergravity on a smooth background. Recall that the Yang-Mills theory arising from D3 branes is dual to string theory on $AdS_5 \times S^5$. The orbifold point of the D1-D5 system can be considered the analogue of free N=4 supersymmetric Yang-Mills theory, since it is the closest we get to a simple theory on the CFT side. Interestingly, it was found [6] that three point functions at large N in weakly coupled 4-d Yang-Mills theory arising from D3 branes were equal to the three point functions arising from the dual supergravity theory, even though the supergravity limit of string theory corresponded to *strongly* coupled Yang-Mills. It would be interesting to ask if there is any similar 'protection' of three point function in the D1-D5 system.

In this paper we find three point functions of a basic class of chiral operators in the orbifold theory. The orbifold group that we have is S^N, the permutation group of N elements. This group is nonabelian, in contrast to the cyclic group Z_N which has been studied more extensively in the past for computation of correlation functions in orbifold

theories [7]. Though there are some results in the literature for general orbifolds [8], the study of nonabelian orbifolds is much less developed than for abelian orbifolds. It turns out however that the case of the S^N orbifolds has its own set of simplifications which make it possible to develop a technique for computation of correlation functions for these theories. In [9] a method was developed to compute the correlation functions for twist operators in the bosonic CFT that emerges from sigma models with such orbifolds. The essential point in that computation was that for the permutation group the following simplification emerges. As in any orbifold theory, one can 'undo' the effect of the twist operators by passing from the space where the theory is defined to a covering Riemann surface where fields are single valued. For orbifolds of the group S^N, the path integral with twist insertions becomes an *unconstrained* path integral with no twist insertions (for one copy of the manifold M) on this covering surface. (Such a simplification would not happen if we pass to the covering surface for a general orbifold group; one would remove the twists by going to the cover but the values of the fields in the path integral at one point on this cover could be constrained to be related to their values at other points on the cover.) The path integral on the covering space was then computed by using the conformal anomaly of the CFT.

In this paper we extend the calculations of [9] to the case of theories with $\mathcal{N} = 4$ supersymmetry. Thus we would obtain in particular results for the above mentioned 'orbifold point' of the D1-D5 system. In order to be able to apply the formalism for any T^4 or $K3$ manifold that can appear in the description of the D1-D5 system, we construct a basic class of chiral operators in the theory in an abstract way, using only the definition of a twist operators of the permutation group, and the form of the superconformal algebra of the $N = 4$ CFT. We then compute explicitly the 3-point function of chiral operators, in the limit of large N, where the surviving contribution comes from the case where the covering surface is a sphere. We observe that the result is significantly simpler than the corresponding result in the bosonic theory.

There are several earlier works that relate to the problem we are studying, in particular [10, 11, 12, 13, 14, 15, 16].

The plan of this paper is as follows. In section 2 we construct the chiral operators corresponding to twist insertions. In section 3 we review briefly the method of [9] for computing correlators of twist insertions, and explain its extension to the supersymmetric case. Section 4 discusses correlation functions of currents that are needed to extend the bosonic computation to the supersymmetric case, and also computes the 2-point functions. In section 5 we compute a simple case of the 3-point functions of these chiral operators, and in section 6 we do the general case. Section 7 is a discussion.

2 Constructing the chiral operators $\sigma_n^\pm$

2.1 Twist operators in the bosonic theory

Let us start with the bosonic theory, and look at the definition of twist operators of the theory. We follow the notation used in [9], where the construction is discussed in more detail. The CFT is defined over the z plane. Over each point z the configuration in the target space is specified by an N-tuple of coordinates $(X_1, \dots X_N)$, where X_i is a collective symbol for the coordinates of a point in the ith copy of the manifold M. The fact that the target space is the orbifold of M^N by the symmetric group S^N means that the point $(X_1, \dots X_N)$ of M^N is to be identified with the points obtained by any permutation of the X_i. First consider the path integral defining the theory in the absence of any twist operators. We can integrate over the X_i independently, without imposing the above identifications, to obtain a partition function that will be $N!$ times the partition function Z that would be obtained if we did take into account the identifications

$$\prod_i \int D[X_i] e^{-S(X_i)} \equiv Z_0^N = N! Z \tag{2.1}$$

where Z_0 is the partition function when the target space is just one copy of M. In the above relation we have assumed that the contribution of the points where two or more of the coordinate sets X_i become equal is of measure zero in the path integral. (Thus note that (2.1) would not be true if the manifold M was replaced by a target space that had a finite number of points; the ignored configurations would then not have measure zero.[1])

To define twist operators consider an element of the permutation group in the form of a 'single cycle' $(1, 2, \dots n)$. (Operators based on a product of two cycles can be regarded as two single cycle operators placed at the same point.) To insert a twist operator for this element of the permutation group at the point $z = 0$ we cut a hole $|z| < \epsilon$ around this point. As we go around this hole counterclockwise, we let the first copy of M go over into the second copy of M, and so on, returning to the first copy after n revolutions. Thus the twist operator changes the boundary conditions around $z = 0$. The correlation function of twist operators σ_i located at points z_i is then defined to be the ratio of the path integral performed with these twisted boundary conditions to the path integral (2.1):

$$\langle \sigma_1^\epsilon(z_1) \dots \sigma_k^\epsilon(z_k) \rangle \equiv \frac{\int_{\text{twisted}} \prod_i D[X_i] e^{-S(X_1 \dots X_N)}}{Z_0^N} \tag{2.2}$$

Note that we have not yet specified the state at the edge of the hole $|z| = \epsilon$. The choice of this state determines which operator we actually insert from the chosen twist sector. The choice of state that gives the lowest dimension operator from the given twist sector will be the one we will call σ_n; other choices of state will give excited states in the same twist sector.

[1] Alternatively we can adopt (2.1) as part of the definition of the theory, taking two copies of the point $(X_1, \dots X_N)$ if $X_1 = X_2$ etc., so that (2.1) is true by construction. When the target space is a manifold the configurations affected are of measure zero.

To specify the state at this edge consider again the operator σ_n inserted at $z = 0$. We pass to local covering space (which we call Σ) by the map which locally looks like

$$z \approx a t^n \tag{2.3}$$

where a is a constant. The n copies of M involved in the twist give rise to a single copy of M on the cover Σ. The hole $|z| < \epsilon$ gives a hole in the t space. We glue in a flat disc into this hole, and extend the path integral perform on Σ to the interior of this disc. This procedure effectively inserts the identity operator at the edge of the hole in the t space, and this gives the lowest dimension operator σ_n in the given twist sector. Note that this procedure inserts the identity operator with a given normalization. The twist operator thus constructed will be called $\sigma_n^\epsilon(0)$; here ϵ is an essential regularization, and will cancel out in final expressions when we compute 3-point functions normalized by the 2-point functions of the operators.

2.2 Fermionic variables

In a supersymmetric theory each copy of M has in addition to the bosonic coordinates X_i a set of fermionic coordinates ψ_i. Upon insertion of a twist operator these fermionic variables from different copies of M are permuted around the twist insertion just like the bosonic variables, and the correlator of twist operators is defined in a manner analogous to (2.2). But we have to take some care in defining the state at the edge of the hole $|z| < \epsilon$. When we pass to the covering space Σ by the map (2.3) the fermionic variables transform as

$$\psi_t(t) = \psi_z(z)(\frac{dz}{dt})^{1/2} = \psi_z(z) a^{1/2} n^{1/2} t^{\frac{n-1}{2}} \tag{2.4}$$

In the z plane we want to have the boundary condition that as we circle the insertion point $z = 0$ counterclockwise we get $\psi_1 \to \psi_2 \to \ldots \to \psi_n \to \psi_1$. While for the spin zero variables X_i this meant that we just have $X \to X$ under transport around $t = 0$ in the t space, we now see that for the spin 1/2 fermionic variables

$$\psi_t(t) \to (-1)^{n-1} \psi_t(t) \tag{2.5}$$

under a counterclockwise rotation in the t space around $t = 0$. We now find a difference between the cases of n odd and n even. The n odd case is simpler, so we discuss it first.

2.3 Twist operators in the supersymmetric theory for odd n

For odd n we see from (2.5) that $\psi_t(t)$ is periodic around $t = 0$, so we can define the state at the edge of the hole just as in the bosonic case, by gluing in a flat disc to close the hole and continuing the fields x, ψ to the interior of the disc. This defines the twist operator $\sigma_n^\epsilon(0)$, just as in the bosonic case.

This operator however has no charge under the R-symmetry group of the $N = 4$ supersymmetric theory. Thus it is not a chiral operator of the theory, since a chiral

operator has $h = j, \bar{h} = \bar{j}$ where $h, \bar{h}$ are the left and right dimensions of the operator and $j, \bar{j}$ are the spins under the left and right $SU(2)$ R-symmetry groups. We therefore seek a natural definition of a chiral operator that is based on the twist operators σ_n.

Each copy of the manifold M gives rise to operators that yield an $N = 4$ supersymmetry algebra, for both the homomorphic and the antiholomorphic variables. Let $J_z^{k,a}(z)$ be the left $SU(2)$ current of the CFT arising from the kth copy of M. The index a takes values 1,2,3. Let

$$J_z^+ = J_z^1 + iJ_z^2, \quad J_z^- = J_z^1 - iJ_z^2 \tag{2.6}$$

The operator

$$J_z^a = \sum_{i=1}^{N} J_z^{i,a}(z) \tag{2.7}$$

is the diagonal element from the set of N $SU(2)$ currents, and gives the left $SU(2)$ current of the orbifold CFT.

Now we note that in the presence of the twist operator $\sigma_n(0)$ we can define the operators

$$J^{+(z)}_{-m/n} \equiv \int \frac{dz}{2\pi i} \sum_{k=1}^{n} \; J_z^{k,+}(z) \; e^{-2\pi i m(k-1)/n} \; z^{-m/n} \tag{2.8}$$

The integral over z is performed over the usual counterclockwise loop around the origin. The integrand is periodic around this loop, since $J_z^{k,+} \rightarrow J_z^{k+1,+}$ due to the cyclic permutation of copies of M around the twist insertion. We have called these operators $J^{+(z)}_{-m/n}$ since they raise the dimension of the twist insertion by m/n. We have included a superscript (z) in these operators to denote the fact that they are operators on the z space; this will distinguish them from modes of the current operators on the t space which we will consider below. These operators raise the $SU(2)$ charge under the diagonal $SU(2)$ by one unit, as can be seen from (2.7) and

$$J_z^3(z_1) J_z^{k,+}(z_2) \sim \frac{1}{z_1 - z_2} J_z^{k,+}(z_2) \tag{2.9}$$

In the t space the operation (2.8) becomes

$$J^{+(z)}_{-m/n} = \int \frac{dz}{2\pi i} \sum_{k=1}^{n} \; J_z^{k,+}(z) \; e^{-2\pi i m(k-1)/n} \; z^{-m/n} \rightarrow \int \frac{dt}{2\pi i} \; J_t^+(t) \; a^{-m/n} t^{-m} \equiv a^{-m/n} \; J^+_{-m} \tag{2.10}$$

In the t space the contour runs around a complete counterclockwise circle around $t = 0$. The n different currents $J_z^{k,+}$ in the z plane give rise to the single current J_t^+ on the covering space Σ, which is the current for the single copy of M that describes the CFT on Σ. To summarize, we find that the operator $J^{+(z)}_{-m/n}$ in the z space gives the operator J^+_{-m} in the t space.

Let us return to the construction of the chiral twist operators. We had observed that the twist operators σ_n which simply permuted copies of M had no charge, and thus had $h > j$ and were not chiral operators. We now ask if we can stay in the same twist

sector, but add other operators to σ_n such that we increase the charge of the resulting operator. Since σ_n had the minimum dimension in its own twist sector, the dimension of the operator will also go up in this process. But if we could achieve $h_{final} = q_{final}$ then we would have constructed a chiral operator.

The operators $J^{+(z)}_{-m/n}$ allow us to make such a construction. We will compute the dimensions of our final chiral operators from the 2-point function in section 4, but it is helpful to use alternative arguments to compute the dimension contributed by various components in the construction of the chiral operator, and we will do that below.

Thus start with σ_n, which is just a twist operator that permutes the copies of M around its insertion point. The dimension of σ_n is

$$\Delta_n = \bar{\Delta}_n = \frac{c}{24}(n - \frac{1}{n}) = \frac{6}{24}(n - \frac{1}{n}) = \frac{1}{4}(n - \frac{1}{n}) \tag{2.11}$$

This dimension can be deduced by looking at the CFT on a cylinder and noting that the twist operator changes a theory based on a set of n separate copies of M to a theory with one single copy of M but on a spatial section that is n times as long. Thus the vacuum energy of the ground state changes from $-nc/24$ to $-(1/n)c/24$, and the change gives the dimension of the twist operator. If M gives an $N = 4$ CFT based on a sigma model with 4 bosons and 4 fermions then we have $c = 6$, and (2.11) follows.

To raise the charge of the operator with minimum increase in dimension consider the application of $J^+_{-1/n}$. The charge goes up by one unit, while the dimension increases by only $1/n$. Note that this low cost in dimension is directly related to the existence of the twist which allows the fractional dimension charge operators (2.8); if we did not have a twist then we could only apply J^+_{-1} which would increase q and h by the same amount, and so not bring an operator with $h > q$ towards an operator with $h = q$.

In the t space we have thus applied J^+_{-1} to the identity operator at $t = 0$; thus we just get the state $J^+_{-1}|0\rangle_{NS} = J^+_t(0)$ where $|0\rangle_{NS}$ is the Neveu–Schwarz vacuum in the t space.

We might try to repeat this process with another application of J^+_{-1} (in the t space), but we find that $J^+_{-1}J^+_{-1}|0\rangle_{NS} = 0$. (This fact and other similar relations used below can be checked by using the commutation relations of the current algebra to find the norm of the state, or more simply by using a bosonic representation of the $N = 4$ algebra and observing that for the manipulations concerned any way of representing the algebra will yield the same results.) We also find that $J^+_{-2}J^+_{-1}|0\rangle_{NS} = 0$. Thus the next step is to construct $J^+_{-3}J^+_{-1}|0\rangle_{NS}$. We keep proceeding in this way, arriving at the operator (in the t space)

$$\sigma^-_n \equiv J^+_{-(n-2)} \ldots J^+_{-3}J^+_{-1}|0\rangle_{NS} \tag{2.12}$$

The dimension of this operator (as seen from the z plane) is

$$h \equiv \Delta^-_n = \Delta_n + \frac{1}{n} + \frac{3}{n} + \ldots + \frac{n-2}{n} = \frac{n-1}{2} \tag{2.13}$$

The charge is

$$q = \frac{n-1}{2} = \Delta^-_n \tag{2.14}$$

and thus the operator is chiral.

The next current operator that we can apply to (2.12) is (in the t space) J^+_{-n}. This raises the charge by one unit, but the dimension in the z plane also rises by one unit, so the resulting operator is another chiral operator:

$$\sigma_n^+ \equiv J^+_{-n} J^+_{-(n-2)} \ldots J^+_{-3} J^+_{-1} |0\rangle_{NS} \tag{2.15}$$

This operator has

$$h \equiv \Delta_n^+ = q = \frac{n+1}{2} \tag{2.16}$$

If we apply the next allowed operator $J^+_{-(n+2)}$ then the charge goes up by one unit but the dimension goes up by $\frac{n+2}{n}$; thus we would get $h > q$ and an operator that is not chiral.

To complete the construction we apply the same steps to the right moving sector using the current $\bar{J}^+$. We thus obtain four chiral operators

$$\sigma_n^{--}, \quad \sigma_n^{+-}, \quad \sigma_n^{-+}, \quad \sigma_n^{++} \tag{2.17}$$

It would appear that we could make other operators in this manner, for example by replacing $\ldots J^+_{-3} J^+_{-1}|0\rangle_{NS}$ by $\ldots J^+_{-2} J^+_{-2}|0\rangle_{NS}$ or $\ldots G^1_{-3/2}\tilde{G}_{2,-3/2} J^+_{-1}|0\rangle_{NS}$. But as we will show in the next section, the operators obtained by the latter constructions are proportional to the operators that we have made above, and so no new operators are obtained this way. There do exist other chiral operators in the theory, which use the details of the structure of the manifold M. For example there are 20 $(1,1)$ forms on K_3 which give rise to chiral operators but only 4 $(1,1)$ forms on T^4. While it should be possible to make and use these additional chiral operators to compute correlation functions with our general methods, we have not done so in this paper. Thus we will consider only the basic operators (2.12), (2.15) (and their counterparts for n even).

2.4 Twist operators in the supersymmetric theory for even n

For n even the construction of the chiral primaries from σ_n is slightly different. The operator σ_n just permutes the copies of M, so the fermionic variables ψ_1 from the first copy cycle around and return to themselves after n rotations around the twist insertion in the z plane. But on the covering surface we see from (2.5) that ψ_t returns to itself after one rotation around $t = 0$ but with a change of sign. This means that we should not close the hole around $t = 0$ by just gluing in a disc and getting the state $|0\rangle_{NS}$ at $t = 0$. Rather we need to insert an operator that creates a Ramond vacuum $|0\rangle_R$ at $t = 0$, so that the fermion ψ_t will indeed be antiperiodic around $t = 0$. We will see that this operator must be a spin field $\mathcal{S}^\alpha, \alpha = \pm$.

We can extract all the relevant properties of this state $|0\rangle_R$ without making any reference to the details of the manifold M, just using the fact that M gives rise to an $N = 4$ supersymmetric theory. The Ramond vacuum can be obtained by a spectral flow

starting from the NS vacuum. We list in Appendix A the relevant formulae for spectral flow. The parameter giving the amount of spectral flow is $\eta = 1$. In the t space we have one copy of M so the CFT has $c = 6$. The NS vacuum has $h = q = 0$. Then we find that the R vacuum has $h = 6/24 = 1/4, q = -1/2$. We thus denote it as $|0^-\rangle_R$. We can act on this state by J_0^+ to obtain another degenerate R vacuum with $h = 1/4, q = 1/2$

$$J_0^+|0^-\rangle_R = |0^+\rangle_R \tag{2.18}$$

The spin fields $\mathcal{S}^\pm$ create the states $|0^\pm\rangle_R$ from $|0\rangle_{NS}$.

Starting with $|0^-\rangle_R$ let us make a state with the maximal charge to dimension ratio, just as we did in the case of n odd above. The first operator we apply is J_0^+. The next lowest dimension operator that we can apply is J_{-2}^+, and so on. We then find the chiral states

$$\sigma_n^- \equiv J^+_{-(n-2)} \dots J^+_{-2} J_0^+ |0^-\rangle_R \tag{2.19}$$

The dimension and charge of this operator (as seen from the z plane) are

$$h \equiv \Delta_n^- = \Delta_n + \frac{1}{4n} + \frac{2}{n} + \dots + \frac{n-2}{n} = \frac{n-1}{2}, \quad q = \frac{n-1}{2} = \Delta_n^- \tag{2.20}$$

Here the contribution Δ_n arises just as in the case of n odd, and $\frac{1}{4n}$ is the contribution to the dimension in the z plane coming from the insertion of the spin field $\mathcal{S}^-$ in the t plane (this field takes $|0\rangle_{NS}$ to $|0^-\rangle_R$). The fact that the dimension $1/4$ in the t plane becomes $1/4n$ in the z plane can be seen from the form of the covering space map $z = at^n$.

We can apply one more current operator to obtain another chiral operator:

$$\sigma_n^+ \equiv J^+_{-n} J^+_{-(n-2)} \dots J^+_{-2} J_0^+ |0^-\rangle_R \tag{2.21}$$

This operator has

$$h \equiv \Delta_n^+ = \frac{n+1}{2}, \quad q = \frac{n+1}{2} \tag{2.22}$$

Applying current operators from the right moving sector in an analogous manner we again obtain four chiral operators of the form (2.17). The charges and dimensions of the operators have the same form in the case n even as in the case n odd.

Notation: The chiral operators constructed from σ_n are denoted $\sigma_n^{\pm\pm}$, and their dimensions are denoted by $\Delta_n^\pm, \bar{\Delta}_n^\pm$. We will also use the notation $\Delta_n = \frac{c}{24}(n - \frac{1}{n}) = \frac{1}{4}(n - \frac{1}{n})$. Δ_n is the contribution to the dimension of the chiral operator from the conformal anomaly.

2.5 Other members of the chiral operator representation

We have constructed states that have $h = q$ and are thus chiral operators of the CFT. By charge conservation, any correlator of chiral operators will vanish. To find nonvanishing

correlators we must look at the $SU(2)$ representation of which the above chiral operator is the highest weight state $|j,m\rangle = |j,j\rangle$. These other states have the form

$$(J_0^-)^k|j,j\rangle = \left(\int \frac{dz}{2\pi i} J_z^-(z)\right)^k |j,j\rangle \tag{2.23}$$

where J_z^- is an element of the diagonal $SU(2)$ (2.7). Thus the operators we study will be given by applications of J_0^+ and J_0^- operators to the twist operators σ_n.

2.6 Universality in the construction of the chiral primaries

We note that the construction of the above chiral operators made no reference to the detailed structure of the manifold M. We used only the fact that the CFT based on M had $N=4$ supersymmetry. Thus M could be a K3 space at a generic point in its moduli space, which is not simply an orbifold of a torus. Thus we are not working with a CFT which can be reduced to a free field theory. In the computations below we will use a bosonic representations of current operators to simplify the calculations; this is allowed because we will be working on a sphere. But it should be noted that these computations could all have been performed by using only the general properties of the $N=4$ algebra.

3 The method of computing correlation functions

In [9] a method was developed to compute the correlation functions for bosonic orbifolds M^N/S^N. We will find that the bosonic result can be extended to obtain the result in the supersymmetric case, after we take into account the current operators that were added to the twist operator σ_n to get the operators σ_n^{--} etc. We review here briefly the method used in [9] and then indicate the way it will be extended to the supersymmetric case.

3.1 The method for bosonic orbifolds

Let us assume that we have only bosonic variables describing the sigma model with target space M. Consider the definition (2.2) of the correlation function of twist operators σ_n. The path integral performed with twisted boundary conditions becomes a path integral on the covering space Σ. The holes at the location of the twist operators are closed by inserting a disc, and thus Σ is a closed surface. In general Σ may have several disconnected components, but we assume here that there is just one component, since all different components can be handled in the same way. The copies of M that are not involved in any of the twists give a contribution to the partition function that cancels out between the numerator and denominator of (2.2).

The genus of the surface Σ depends on the orders of the twist operators. At the insertion of the operator $\sigma_{n_j}(z_j)$ the covering surface Σ has a branch point of order n_j, which means that n_j sheets of Σ meet at z_j. One says that the ramification order at z_j is

$r_j = n_j - 1$. Suppose further that over a generic point z here are s sheets of the covering surface Σ. Then the genus g of Σ is given by

$$g = \frac{1}{2}\sum_j r_j - s + 1 \tag{3.1}$$

The path integral over the z plane with twist insertions becomes a path integral for a CFT based on only one copy of M, on the surface Σ with no twist insertions or any other operator insertions. But the metric to be used on Σ to compute the path integral is the metric induced from the z plane, and it is the dependence of the path integral on this metric that encodes the dependence of the correlation function on the location of twist operators in the z plane. We can compute the path integral for some fiducial metric $\tilde{g}$ on Σ, provided we take into account the correction due to the metric change by using the conformal anomaly. If $ds^2 = e^{\phi} d\tilde{s}^2$, then the partition function $Z^{(s)}$ computed with the metric ds^2 is related to the partition function $Z^{(\tilde{s})}$ computed with $d\tilde{s}^2$ through

$$Z^{(s)} = e^{S_L} Z^{(\tilde{s})} \tag{3.2}$$

where

$$S_L = \frac{c}{96\pi}\int d^2t\sqrt{-g^{(\tilde{s})}}[\partial_\mu\phi\partial_\nu\phi g^{(\tilde{s})\mu\nu} + 2R^{(\tilde{s})}\phi] \tag{3.3}$$

is the Liouville action [17]. Here c is the central charge of the CFT for one copy of M.

We can choose the fiducial metric $\tilde{g}$ to be the flat metric $dtd\bar{t}$ everywhere except for a set of isolated points. Then the metric induced from the z space is written as

$$ds^2 = dzd\bar{z} = e^{\phi} dtd\bar{t}. \tag{3.4}$$

Let us call the part of Σ that excludes these isolated curvature points and the punctures arising from the twist insertions as the 'regular region' of Σ. It was shown in [9] that for the cases that we are interested in the contribution to the path integral from these excluded points is zero, and thus we only have the contribution $S_L^{(1)}$ from the regular region. Since the metric is flat by construction in this region we get no contribution from the term $R^{(\tilde{s})}\phi$ in the action (3.3). Thus we have

$$S_L = S_L^{(1)} = \frac{1}{96\pi}\int d^2t[\partial_\mu\phi\partial^\mu\phi] \tag{3.5}$$

We rewrite (3.5) as

$$S_L = -\frac{1}{96\pi}\int d^2t[\phi\partial_\mu\partial^\mu\phi] + \frac{1}{96\pi}\int_{\partial\Sigma}\phi\partial_n\phi \tag{3.6}$$

Here ∂_n is the normal derivative at the boundaries of Σ. From (3.4) we find that

$$\phi = \log[\frac{dz}{dt}] + \log[\frac{d\bar{z}}{d\bar{t}}] \tag{3.7}$$

so that

$$\partial_\mu \partial^\mu \phi = \partial_t \partial_{\bar{t}} \phi = 0 \tag{3.8}$$

and we get

$$S_L = \frac{1}{96\pi} \int_{\partial\Sigma} \phi \partial_n \phi \tag{3.9}$$

Thus we get the desired correlation function of twist insertions on the z plane as a sum of contributions from local expressions (3.9) from a finite set of points on Σ. An essential (and generally nontrivial) step in the calculation is finding the map $t(z)$ that gives the branched cover of the z plane with the given ramifications at the insertions of the twist operators. In this way any correlation function of twist operators on the z plane for the theory M^N/S^N can be deduced from a knowledge of the partition functions Z_g for the theory for one copy of M on Riemann surfaces of different genera g.

The final primary fields of the CFT are not the twists σ_n but operators O_n which are made from $\sigma_n = \sigma_{(i_1 \ldots i_n)}$ by symmetrizing over all different ways in which the n indices i_k involved in σ_n can be chosen from the set of indices $i = 1 \ldots N$ which denote all the available copies of M. The correlations functions of the O_n are therefore just given by combinatorial factors multiplying correlators of the σ_n. Note that when we look at different choices of indices making up the O_n then we can get a finite number of different covering surfaces Σ. But we find from the combinatorics that if N is large then for a given choice of orders of twist operators the leading contribution comes from the case when Σ is a sphere [9]. The contribution from the cases where Σ is higher genus is suppressed by a relative factor $1/N^g$.

3.2 The supersymmetric case

The above result on combinatorics applies as much to the supersymmetric case as to the bosonic case. Thus we concentrate on the case where Σ is a sphere in the present paper; this will give us the leading order result at large N.

Let us consider the 3-point function of twist operators in the supersymmetric theory. As we saw in the previous section, the added complication in the supersymmetric case is that on Σ we should not just close the puncture at the location of the twist operator by inserting the identity state $|0\rangle_{NS}$. Rather, we have to insert current operators at these locations, and for even n we also need an operator - the 'spin field'- that takes the state $|0\rangle_{NS}$ to $|0^-\rangle_R$. Thus Σ is found in the same way as in the case of the bosonic orbifold, but instead of computing the path integral on Σ we need to compute a correlation function on Σ.

As in the bosonic case (2.1), let Z_0 be the partition function for the supersymmetric theory when the target space is one copy of M

$$Z_0 = \int_g D[X, \psi] e^{-S[X,\psi]} \tag{3.10}$$

The partition function is computed in some chosen metric g on the z space. (In [9] the z space was chosen to be a large flat disc which was closed to a sphere by gluing an identical disc at its boundary.)

Choosing the operators to be σ_n^{--} for concreteness, we define the correlation function analogous to (2.2), but with insertions of currents J^+, J^- required in the construction of the chiral operator:

$$\begin{aligned}
&\langle\sigma_1^{\epsilon--}(z_1)\ldots\sigma_k^{\epsilon--}(z_k)\rangle\\
&\equiv \frac{1}{Z_0^s}\int_{\text{twisted}}\prod_{m=1}^{s}D[X_m,\psi_m]e^{-S(X_1\ldots X_N,\psi_1\ldots\psi_N)}\prod_{i,j}\int\frac{dq_{ij}}{2\pi i}J^{\pm}(q_{ij})(q_{ij}-z_j)^{-n_{ij}/n_j}\\
&\equiv \frac{\mathcal{Q}}{Z_0^s}
\end{aligned} \tag{3.11}$$

We have assumed that s copies of M are joined by the twisted boundary conditions, so that the path integral over the remaining $N-s$ copies of M cancels out between the numerator and denominator in (3.11). The q_{ij} are integrated over contours around the z_j, and the integers n_{ij} are given by the form of the chiral operators discussed in the last section.

Passing to the covering space Σ, which we are assuming to be a sphere, we get the path integral for the theory with one copy of M but with a metric induced from the z space

$$\begin{aligned}
\mathcal{Q} &= \int_{g_{induced}}D[X,\psi]e^{-S[X,\psi]}\prod_{i,j}\int J_t^{\pm}(q_{ij})\frac{dq_{ij}}{2\pi i}(q_{ij}-t_j)^{-n_{ij}}\prod_i\mathcal{S}^-(t_i)\\
&= e^{S_L^\epsilon}\int_g D[X,\psi]e^{-S[X,\psi]}\prod_{i,j}\int J_t^{\pm}(q_{ij})\frac{dq_{ij}}{2\pi i}(q_{ij}-t_j)^{-n_{ij}}\prod_i\mathcal{S}^-(t_i)\\
&= e^{S_L^\epsilon}(\int_g D[X,\psi]e^{-S[X,\psi]})\ \langle\prod_{i,j}\int J_t^{\pm}(q_{ij})\frac{dq_{ij}}{2\pi i}(q_{ij}-t_j)^{-n_{ij}}\prod_i\mathcal{S}^-(t_i)\rangle\\
&= e^{S_L^\epsilon}Z_0\ \langle\prod_{i,j}\int J_t^{\pm}(q_{ij})\frac{dq_{ij}}{2\pi i}(q_{ij}-t_j)^{-n_{ij}}\prod_i\mathcal{S}^-(t_i)\rangle
\end{aligned} \tag{3.12}$$

Here $\mathcal{S}^-$ are the spin field insertions for even n operators. S_L^ϵ is the Liouville action arising from the conformal anomaly when we rewrite the path integral for the metric $g_{induced}$ (which is induced from the z space onto the t space) in terms of the fiducial metric g on the sphere that was used to define Z_0 in (3.10). S_L^ϵ depends upon the cutoffs used in defining the twist operators.

The 3-point function normalized by the 2-point functions is

$$\begin{aligned}
&\langle\sigma_1^{--}(z_1)\sigma_2^{--}(z_2)\sigma_3^{--\dagger}(z_3)\rangle\\
&\equiv \frac{\langle\sigma_1^{\epsilon--}(z_1)\sigma_2^{\epsilon--}(z_2)\sigma_3^{\epsilon--\dagger}(z_3)\rangle}{\langle\sigma_1^{\epsilon--}(0)\sigma_1^{\epsilon--\dagger}(1)\rangle^{1/2}\langle\sigma_2^{\epsilon--}(0)\sigma_2^{\epsilon--\dagger}(1)\rangle^{1/2}\langle\sigma_3^{\epsilon--}(0)\sigma_3^{\epsilon--\dagger}(1)\rangle^{1/2}}
\end{aligned} \tag{3.13}$$

The power of Z_0 cancels out in (3.13), after we note the relation (3.1) and the fact that a correlator of the form $\langle\sigma_n\sigma_n^\dagger\rangle$ is proportional to Z_0^{-n}. Let us denote the Liouville action for the 3-point function by $S_L[\sigma_1\sigma_2\sigma_3]$ and for the 2-point functions by $S_L[\sigma_1\sigma_1^\dagger]$ etc. Let us also write the correlator of currents and spin fields in (3.12) as $\langle J,\mathcal{S}\rangle_{\sigma_1\ldots\sigma_k}$. Then we have

$$\begin{aligned}&\langle\sigma_1^{--}(z_1)\sigma_2^{--}(z_2)\sigma_3^{--\dagger}(z_3)\rangle\\&=e^{S_L[\sigma_1\sigma_2\sigma_3^\dagger]-\frac{1}{2}S_L[\sigma_1\sigma_1^\dagger]-\frac{1}{2}S_L[\sigma_2\sigma_2^\dagger]-\frac{1}{2}S_L[\sigma_3\sigma_3^\dagger]}\ \frac{\langle J,\mathcal{S}\rangle_{\sigma_1\sigma_2\sigma_3^\dagger}}{\langle J,\mathcal{S}\rangle^{\frac{1}{2}}_{\sigma_1\sigma_1^\dagger}\langle J,\mathcal{S}\rangle^{\frac{1}{2}}_{\sigma_2\sigma_2^\dagger}\langle J,\mathcal{S}\rangle^{\frac{1}{2}}_{\sigma_3\sigma_3^\dagger}}\end{aligned}\tag{3.14}$$

The contribution of the Liouville terms in the above equation was shown in [9] to be the three point function of twist operators in the bosonic orbifold theory, where the central charge was set to $c = 6$. (This is discussed in more detail below.) The computation of 3-point functions in the supersymmetric theory then just reduces, by (3.14), to a computation of correlation functions of currents and spin fields on the covering surface Σ, for the 3-point function and for the 2-point functions in the denominator. The correlation functions of currents can of course be computed in any metric on Σ, since they are independent of the metric.

If all the chiral operators have odd n twist operators then the only correlation functions that we need are correlators of current operators on Σ. The other possible nonzero three point function is where two of the operators have even n twists. In both these cases the correlation functions (for Σ a sphere) can be computed purely in terms of the properties of the chiral algebra of the $N = 4$ supersymmetric theory. (We can 'undo' a contour of J^+ from around one point on Σ and replace it by contours surrounding the other points in the correlator. The contour moves freely through any other operators J^+ on Σ, while it picks up a contribution from locations of J^- operators determined purely by the $N = 4$ algebra.) Given this fact we can use any convenient representation of this algebra without losing generality in the choice of M. We use a representation of current algebra in terms of free bosonic fields; the current operators are represented by exponentials and polynomials in these new bosonic variables. The spin fields are represented by exponentials in these bosons as well.

We will normalize the current and spin field insertions in the next section such that

$$\langle J,\mathcal{S}\rangle_{\sigma_i\sigma_i^\dagger}=1\tag{3.15}$$

It will also be convenient to use the notation

$$\sigma_n^{--}(z)\equiv\frac{\sigma_n^{\epsilon--}(z)}{\langle\sigma_n^{\epsilon--}(0)\sigma_n^{\epsilon--\dagger}(1)\rangle^{1/2}}\tag{3.16}$$

It is convenient to compute the fusion coefficients (and thus the 3-point functions) by computing

$$\frac{\langle\sigma_1^{--}(0)\sigma_2^{--}(a)\sigma_3^{--\dagger}(\infty)\rangle}{\langle\sigma_3^{--}(0)\sigma_3^{--\dagger}(\infty)\rangle}\equiv|C_{1,2,3}^{---}|^2|a|^{-2(\Delta_1^-+\Delta_2^--\Delta_3^-)}\tag{3.17}$$

From (3.14) and (B.4) we see that the LHS of (3.17) is

$$\frac{\langle\sigma_1^{--}(0)\sigma_2^{--}(a)\sigma_3^{--\dagger}(\infty)\rangle}{\langle\sigma_3^{--}(0)\sigma_3^{--\dagger}(\infty)\rangle} = |C_{1,2,3}|^{12}\ |a|^{-2(\Delta_1+\Delta_2-\Delta_3)}\ \frac{\langle J,\mathcal{S}\rangle_{\sigma_1(0)\sigma_2(a)\sigma_3^\dagger(\infty)}}{\langle J,\mathcal{S}\rangle_{\sigma_3(0)\sigma_3^\dagger(\infty)}} \tag{3.18}$$

where we have used (3.15). Here $C_{1,2,3}$ is the fusion coefficient (computed in [9]) of twist fields for a bosonic theory with $c = 1$.

To summarize, the three point functions of the supersymmetric theory will be computed as the product of two contributions. The first part is the contribution of the conformal anomaly, which arises in the map of the z sphere to the t sphere. This part of the calculation is identical to that for the bosonic case, after we note that the central charge for the theory based on one copy of M is $c = 6$ for the field content of a $N = 4$ supersymmetric theory. The second part is the correlator of current operators and spin fields on Σ; these fields will all be represented by exponentials and polynomials of bosons on Σ.

4 The contribution of the current insertions

4.1 Representing the current algebra by free fields

As mentioned above when Σ is a sphere we can use the following simplification. We need only the OPEs of the chiral algebra generators to get the correlation function, and so we can represent these generators in any manner that reproduces these OPEs.

Let us therefore take a specific example of a system with $\mathcal{N} = 4$ superconformal symmetry, and use it to construct the chiral algebra generators and their correlation functions on the sphere. We take two complex bosons (X_1 and X_2) and two complex fermions (Ψ_1 and Ψ_2). The elements of the superconformal algebra for such system are given by:

$$J^a(z) = \frac{1}{2}\Psi_1^\dagger\sigma^a\Psi_1 + \Psi_2^\dagger\sigma^a\Psi_2, \qquad T(z) = \partial X_i^\dagger\partial X_i + \frac{1}{2}\Psi_i^\dagger\partial\Psi_i - \frac{1}{2}\partial\Psi_i^\dagger\Psi_i \tag{4.1}$$

$$G^1(z) = \sqrt{2}\Psi_2^\dagger\partial X_1 - \sqrt{2}\Psi_1\partial X_2, \quad G^2(z) = \sqrt{2}\Psi_1^\dagger\partial X_1 + \sqrt{2}\Psi_2\partial X_2, \tag{4.2}$$

$$\tilde{G}_1(z) = \sqrt{2}\Psi_2\partial X_1^\dagger - \sqrt{2}\Psi_1^\dagger\partial X_2^\dagger, \quad \tilde{G}_2(z) = \sqrt{2}\Psi_1\partial X_1^\dagger + \sqrt{2}\Psi_2^\dagger\partial X_2^\dagger. \tag{4.3}$$

Let us introduce the real components of the bosonic fields:

$$X_1 = \frac{\phi_1 + i\phi_2}{\sqrt{2}}, \qquad X_2 = \frac{\phi_3 + i\phi_4}{\sqrt{2}} \tag{4.4}$$

and bosonize the fermions:

$$\Psi_1 = e^{i\phi_5}, \qquad \Psi_2 = e^{i\phi_6}. \tag{4.5}$$

One can now rewrite the superconformal generators in terms of six real bosonic fields ϕ_i:

$$\begin{aligned} J^3(z) &= \frac{i}{2}\left(\partial\phi_5 - \partial\phi_6\right), \qquad J^+(z) = \exp(i\phi_5 - i\phi_6), && (4.6)\\ J^-(z) &= \exp(-i\phi_5 + i\phi_6) \qquad T(z) = \frac{1}{2}\sum_{i=1}^{6}\partial\phi_i^\dagger\partial\phi_i && (4.7)\\ G^1(z) &= \exp(-i\phi_6)(\partial\phi_1 + i\partial\phi_2) - \exp(i\phi_5)(\partial\phi_3 + i\partial\phi_4), && (4.8)\\ G^2(z) &= \exp(-i\phi_5)(\partial\phi_1 + i\partial\phi_2) + \exp(i\phi_6)(\partial\phi_3 + i\partial\phi_4) && (4.9) \end{aligned}$$

The components of $\tilde{G}$ can be obtained from G^a by taking a complex conjugate.

For later use it will be convenient to adopt a notation where all 6 bosons (4 original bosons and 2 from bosonizing the fermions) are grouped together into a vector ϕ_a. Then we write

$$\begin{aligned} J^3(z) &= \frac{i}{2}\sum_j e_a\partial_z\phi_j^a(z), && (4.10)\\ J^+(z) &= \sum_j \exp\left(ie_a\phi_j^a(z)\right), \qquad J^-(z) = \sum_j \exp\left(-ie_a\phi_j^a(z)\right), && (4.11)\\ G^1(z) &= \sum_j \left\{\exp\left(-ic_a\phi_j^a(z)\right)A_b\partial\phi_j^b(z) - \exp\left(id_a\phi_j^a(z)\right)B_b\partial\phi_j^b(z)\right\}, && (4.12)\\ G^2(z) &= \sum_j \left\{\exp\left(ic_a\phi_j^a(z)\right)B_b\partial\phi_j^b(z) + \exp\left(-id_a\phi_j^a(z)\right)A_b\partial\phi_j^b(z)\right\}. && (4.13) \end{aligned}$$

Here we have

$$\begin{aligned} \mathbf{A} = (1, i, 0, 0, 0, 0), \quad \mathbf{B} = (0, 0, 1, i, 0, 0),\\ \mathbf{c} = (0, 0, 0, 0, 0, 1), \quad \mathbf{d} = (0, 0, 0, 0, 1, 0) \end{aligned} \qquad (4.14)$$

Starting from the general form (4.10)–(4.13) and requiring that the elements of the chiral algebra satisfy their OPEs, we get the following constraints on real vectors $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ and complex vectors $\mathbf{A}$, $\mathbf{B}$:

$$\mathbf{e}\cdot\mathbf{e} = 2, \qquad \mathbf{c}\cdot\mathbf{e} = -1, \qquad \mathbf{d}\cdot\mathbf{e} = 1, \qquad \mathbf{A}\cdot\mathbf{A}^* = \mathbf{B}\cdot\mathbf{B}^* = 2, \qquad (4.15)$$

$$\mathbf{c}\cdot\mathbf{d} = \mathbf{e}\cdot\mathbf{A} = \mathbf{e}\cdot\mathbf{B} = \mathbf{c}\cdot\mathbf{A} = \mathbf{c}\cdot\mathbf{B} = \mathbf{d}\cdot\mathbf{A} = \mathbf{d}\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{A} = \mathbf{B}\cdot\mathbf{B} = \mathbf{A}^*\cdot\mathbf{B} = 0.$$

$$\mathbf{e} = \mathbf{d} - \mathbf{c}, \qquad \mathbf{d}\cdot\mathbf{d} = \mathbf{c}\cdot\mathbf{c} = 1. \qquad (4.16)$$

These are the only properties that we will explicitly use.

It will be convenient not to use vectors $\mathbf{d}$ and $\mathbf{c}$, but use instead $\mathbf{e}$ and

$$\mathbf{f} = \mathbf{c} + \mathbf{d}: \qquad \mathbf{f}\cdot\mathbf{f} = 2, \qquad \mathbf{f}\cdot\mathbf{e} = 0 \qquad (4.17)$$

In this representation through free bosons it is easy to establish the claim made in the previous section about the possible chiral operators in the CFT. For odd n, the operators

σ_n^- had the form $J^+_{n-2}\dots J^+_{-3}J^+_{-1}|0\rangle_{NS}$. The same dimension and charge can be obtained by replacing $J^+_{-3}J^+_{-1}$ by $J^+_{-2}J^+_{-2}$ or by $G^1_{-3/2}\tilde{G}_{2,-3/2}J^+_{-1}$. But as we argue now, these other operators will all be proportional to the original operator, and so there will be no further chiral operators with this charge and dimension in the twist sector of σ_n.

This operator has charge $\frac{n-1}{2}$ under J^3 and a dimension $\frac{n-1}{2}$. If we take any combination of the generators $T, G^\alpha, \tilde{G}^\alpha J^a$ from (4.10)–(4.13) then we will get expressions which are of the form 'exponential in the ϕ_a' times 'polynomial in the ϕ_a'. But the required dimension and charge are obtained only by the expression

$$|p\rangle = b^{-p^2/n} :\exp\left(ipe_a\Phi^a(0)\right): , \qquad p=\frac{n-1}{2}. \tag{4.18}$$

Here $z\approx bt^n$ is the map taking the operator defines in the z plane to its image in the t space. We will deduce the power of b in (4.18) below, but first let us look at the charge and dimension of the operator. The charge determines the exponential, and adding any polynomial or any exponential in a direction orthogonal to the vector e_a increases the dimension without increasing the charge. This establishes the above claim that there is a unique operator with the desired quantum numbers. Note that using the free boson representation does not limit us to working with a free CFT; the same result could be proven by computing the determinant of the matrix of dot products between the above mentioned states, and finding that the matrix has rank 1 after use of the commutation relations of the chiral algebra.

Let us now obtain (4.18) directly from the definition of the chiral operator in the z plane, and thus obtain the required power of b. Let us take n odd. We have from (2.10)

$$J^{+(z)}_{-1/n}\sigma_n(0) = b^{-1/n}\int\frac{dt}{2\pi i}J^+_t(t)t^{-1} = b^{-1/n}J^+_t(0) = b^{-1/n}\exp\left(ie_a\Phi^a(0)\right) \tag{4.19}$$

(Here and in what follows we will identify the operator in the z plane with its representation in the t space, letting it be clear from the notation which representation we are referring to.)

If we apply another current operator then we get

$$J^{+(z)}_{-k/n}J^{+(z)}_{-1/n}\sigma_n(0) = b^{-k/n}b^{-1/n}\int\frac{dt}{2\pi i}t^{-k}\exp\left(ie_a\Phi^a(t)\right)\exp\left(ie_a\Phi^a(0)\right) \tag{4.20}$$

From the OPE $\exp\left(ie_a\Phi^a(t)\right)\exp\left(ie_a\Phi^a(0)\right)\sim t^2$ we see the claim made earlier that the lowest allowed value of k is 3. For $k=3$ we get

$$J^{+(z)}_{-3/n}J^{+(z)}_{-1/n}\sigma_n(0) = b^{-4/n}\exp\left(2ie_a\Phi^a(0)\right) \tag{4.21}$$

for the operator in the t space.

Proceeding in this manner we find that

$$J^{+(z)}_{-(n-2)/n}\dots J^{+(z)}_{-3/n}J^{+(z)}_{-1/n}\sigma_n(0) = b^{-p^2/n}\exp\left(ipe_a\Phi^a(0)\right), \quad p=\frac{n-1}{2} \tag{4.22}$$

In a similar manner we can compute other sets of operators that would yield chiral states. Let

$$z \approx bt^n(1+\xi t+O(t^2)). \tag{4.23}$$

be the map from t plane to the z plane near point $t=0$. Then

$$\begin{aligned} J^{+(z)}_{-2/n}J^{+(z)}_{-2/n}\sigma_n(0) &= J^{+(z)}_{-2/n}b^{-2/n} : (ie_b\partial\Phi^b(0)-\frac{2\xi}{n})\exp\left(ie_a\Phi^a(0)\right) : \\ &= 2b^{-4/n} : \exp\left(2ie_a\Phi^a(0)\right) : \end{aligned} \tag{4.24}$$

so we get the same state as we would get from $J^{+(z)}_{-3/n}J^{+(z)}_{-1/n}\sigma_n(0)$. (We had argued above that such had to be the case from a consideration of the charges and dimensions of the chiral operators.) Similarly we find that

$$\tilde{G}_{2,-3/2}G^1_{-3/2}J^+_{-1}\sigma_n(0) \sim b^{-4/n} : \exp\left(2ie_a\Phi^a(0)\right) : \tag{4.25}$$

Let us now consider the case of even n. Now we need to insert a spin field in the t space to obtain the vacuum $|0^-\rangle_R$ from $|0\rangle_{NS}$. We can reproduce the charge, dimension and OPEs of the spin field by using the operator $\exp\left(\pm\frac{1}{2}ie_a\Phi^a(t)\right)$ to represent the spin fields $\mathcal{S}^\pm$. Again using the fact that the chiral operators have dimension equal to charge, we find that

$$J^{+(z)}_{-(n-2)/n}\dots J^{+(z)}_{-2/n}J^{+(z)}_0\sigma_n(0) = b^{-p^2/n}\exp\left(ipe_a\Phi^a(0)\right), \quad p=\frac{n-1}{2} \tag{4.26}$$

is the unique state giving the chiral operator with $h=q=\frac{n-1}{2}$ in the twist sector of σ_n. Similarly, σ^+_n is given by taking in (4.26) the value $p=\frac{n+1}{2}$.

Note that we will be using the correlators of at most two spin fields on a sphere, and for such correlators the properties of the spin field encoded in this bosonic representation entails no loss of generality. In particular this representation is not equivalent to assuming that we are dealing with a theory that can be reduced to free bosons.

4.2 Two point functions.

We now wish to compute the 2-point functions of twist operators, for two reasons. First, we extract the dimensions of the operators from the 2-point functions. Second, to define the 3-point function we have to know the normalization of the operators that go into the computation, and this is done by choosing the normalizations of the 2-point functions of the operators with their own conjugates.

We have seen that in the representation through free bosons the chiral operators take, when lifted to the t sphere, the form (4.18)

$$|p\rangle = Ab^{-p^2/n} : \exp\left(ipe_a\Phi^a(0)\right) : \tag{4.27}$$

where $p=\frac{n\pm1}{2}$ for $\sigma^\pm_n$. While the structure of the map $z(t)$ fixes the b dependence in the last equation, the overall normalization constant A has not been determined so far. To

do so, one needs to evaluate the two point function of such exponentials, and we will do this below.

Let us consider the operator $\sigma_n^{\epsilon--}$. We place this operator at $z=0$ and its conjugate $\sigma_n^{\epsilon--\dagger}$ at $z=a$. Note that if σ_n^- corresponds to twist $(1,\ldots,n)$, and has charge $\frac{n-1}{2}$, then $\sigma_n^{-\dagger}$ corresponds to the twist $(n,\ldots,1)$ and has a charge $-(n-1)/2$. Following the method outlined in the previous section, we go from the z sphere to the t sphere by the map [9]:

$$z=\frac{at^n}{t^n-(t-1)^n} \tag{4.28}$$

The path integral in the presence of the chiral fields then becomes, on the t sphere, a path integral with the insertion of exponentials, but with no twists. The metric on the t sphere is to be induced from the z sphere, but we use a fiducial sphere metric for t while taking into account the change of metric by the conformal anomaly. Using the method outlined in section 3 (equation (3.11), (3.12)) we get

$$\langle\sigma_n^{\epsilon--}(0)\sigma_n^{\epsilon--\dagger}(a)\rangle \;=Z_0^{1-n}e^{S_L^\epsilon}\langle\frac{n-1}{2},z=a|\frac{n-1}{2},z=0\rangle, \tag{4.29}$$

where $\langle\frac{n-1}{2},z=a|\frac{n-1}{2},z=0\rangle$ is a correlation function of two exponentials[2] (4.27):

$$\begin{aligned}&\langle\frac{n-1}{2},z=a|\frac{n-1}{2},z=0\rangle=|A|^2\times\\ &\Big\langle|b_0|^{-(n-1)^2/2n}:\exp\Big(i\frac{n-1}{2}e_a\Phi^a(0)\Big):|b_1|^{-(n-1)^2/2n}:\exp\Big(-i\frac{n-1}{2}e_a\Phi^a(1)\Big):\Big\rangle\\ &\quad=|A|^2|b_0|^{-(n-1)^2/2n}|b_1|^{-(n-1)^2/2n}.\end{aligned} \tag{4.30}$$

The values of b_0 and b_1 in (4.30) are determined by the asymptotic behavior of (4.28) near $t=0$ and $t=1$:

$$z=a(-1)^{n-1}t^n+O(t^{n+1}),\qquad b_0=a; \tag{4.31}$$
$$z=a+a(t-1)^n+O((t-1)^{n+1}),\qquad b_1=a. \tag{4.32}$$

Taking this into account, we get an expression for the correlator of two exponents:

$$\langle\frac{n-1}{2},z=a|\frac{n-1}{2},z=0\rangle=|A|^2|a|^{-(n-1)^2/n}. \tag{4.33}$$

Note that in (3.14) the following shorthand notation has been used for the same two point function:

$$\langle J,\mathcal{S}\rangle_{\sigma_n^{--}\sigma_n^{--\dagger}}\equiv\langle\frac{n-1}{2},z=1|\frac{n-1}{2},z=0\rangle. \tag{4.34}$$

Let us now choose

$$A=1,\quad |p\rangle=b^{-p^2/n}:\exp\left(ipe_a\Phi^a(0)\right): \tag{4.35}$$

[2] To make this expression more compact, we use Φ to represent the *total* bosonic field, while in the rest of this paper Φ represents only holomorphic part of such field.

In this case the expression (4.34) equals unity, and we also get unity for the denominator in (3.14).

Let us now go back to the complete two point function (4.29). The contribution of conformal anomaly was evaluated in [9] and it has the following form:

$$Z_0^{1-n} e^{S_L^\epsilon} = B_\epsilon |a|^{-4\Delta_n}. \tag{4.36}$$

Here B_ϵ is a coefficient which depends on regularization parameters, but not on a.

Then for the two point function we finally get:

$$\langle \sigma_n^{\epsilon--}(0) \sigma_n^{\epsilon--\dagger}(a) \rangle = B_\epsilon |a|^{-4\Delta_n} |a|^{-(n-1)^2/n} = B_\epsilon |a|^{-2(n-1)} \tag{4.37}$$

Thus the dimension of the operator σ_n^{--} is $\Delta_n^- = \bar{\Delta}_n^- = \frac{n-1}{2}$.

The dimensions of the other operators $\sigma_n^{+-}, \sigma_n^{-+}, \sigma_n^{++}$ can be computed in a similar manner.

5 Three point function of the form $\langle \sigma_n^\pm \sigma_m^\pm \left(\sigma_{m+n-1}^\pm\right)^\dagger \rangle$

To start with we will look at a special class of 3-point functions:

$$\langle \sigma_n^\pm(0) \sigma_m^\pm(a) \left(\sigma_{m+n-1}^\pm\right)^\dagger (z) \rangle. \tag{5.1}$$

The fact that we have written only one superscript $\pm$ on each σ_n indicates that we are looking at only the holomorphic part of the 3-point function. The $\dagger$ operation changes a permutation $(1, 2, \ldots n)$ to the inverse element of the permutation group $(n, n-1, \ldots 1)$, and also changes the sign of the J_3 charge of the operator. If the permutations of order n and order m combine into a permutation with a single nontrivial cycle, then the maximal order of this cycle is $n + m - 1$, and this is the order of permutation chosen for the third operator in the above expression. The value of this subclass of correlators was computed by a recursion argument in [13], but we reproduce the result by our method as a warmup towards the general case which we study in the next section.

Note that σ_n^- has charge $\frac{n-1}{2}$ while σ_n^+ has charge $\frac{n+1}{2}$. Thus by charge conservation

$$\begin{aligned}
\langle \sigma_n^+(0) \sigma_m^+(a) \left(\sigma_{m+n-1}^+\right)^\dagger (z) \rangle &= 0 \\
\langle \sigma_n^+(0) \sigma_m^-(a) \left(\sigma_{m+n-1}^-\right)^\dagger (z) \rangle &= 0 \\
\langle \sigma_n^-(0) \sigma_m^-(a) \left(\sigma_{m+n-1}^+\right)^\dagger (z) \rangle &= 0.
\end{aligned}$$

The nonvanishing fusion coefficients will be denoted as

$$\sigma_n^-(0) \sigma_m^-(a) = a^{\Delta_{m+n-1}^- - \Delta_n^- - \Delta_m^-} C_{n,m,n+m-1}^{---} \sigma_{n+m-1}^-(0), \tag{5.2}$$

$$\sigma_n^+(0) \sigma_m^-(a) = a^{\Delta_{m+n-1}^+ - \Delta_n^+ - \Delta_m^-} C_{n,m,n+m-1}^{+-+} \sigma_{n+m-1}^+(0), \tag{5.3}$$

$$\tag{5.4}$$

To compute the fusion coefficients we use (3.18). In the expression (3.18) we have the factors coming from the conformal anomaly

$$|C_{1,2,3}|^{12}\ |a|^{-2(\Delta_1+\Delta_2-\Delta_3)} = a^{-\frac{1}{4}(1-\frac{1}{n}-\frac{1}{m}+\frac{1}{n+m-1})}(C_{n,m,n+m-1})^6, \tag{5.5}$$

$$\log|C_{n,m,m+n-1}|^2 = -\frac{1}{12}\left(n+\frac{1}{n}\right)\log n - \frac{1}{12}\left(m+\frac{1}{m}\right)\log m \tag{5.6}$$
$$+\frac{1}{12}\left(q+\frac{1}{n}+\frac{1}{m}-1\right)\log q - \frac{1}{12}\left(1+\frac{1}{q}-\frac{1}{n}-\frac{1}{m}\right)\log\left(\frac{(q-1)!}{(m-1)!(n-1)!}\right).$$

Here $q = m+n-1$. The power of a in (5.5) arises from the dimensions of the bosonic twists σ_n which are $\Delta_n = \frac{c}{24}(n-\frac{1}{n})$.

The remainder of the contribution to the OPE comes from the insertions of J^+ (and spin fields for even n) at the images the twist operators on Σ. We thus have to compute a correlation function of these elements of the chiral algebra, and since we are on the sphere, we lose no generality by using a representation of the currents in terms of free bosons. In the previous section we have shown (eq. (4.35)) that the twist operator $\sigma_n^{\pm}(0)$ gives rise to the following insertion on the t sphere:

$$|p\rangle = b_0^{-p^2/n} : \exp\left(ipe_a\Phi^a(0)\right) :, \tag{5.7}$$

where $p = \frac{n-1}{2}$ for σ_n^- and $p = \frac{n+1}{2}$ for σ_n^+. The value of b_0 is given by the leading behavior of the map near $t=0$:

$$z = b_0 t^n + O(t^{n+1}). \tag{5.8}$$

Consider now a second insertion at the point $z=a$ (which maps to $t=1$):

$$|q\rangle = b_1^{-q^2/m} : \exp\left(iqe_a\Phi^a(1)\right) : \tag{5.9}$$

with $q = \frac{m\pm1}{2}$, depending on the charge of $\sigma_m^{\pm}(a)$. To evaluate b_0, b_1 we recall the map used in [9] to go from the z sphere to the t sphere. This map was constructed in terms of the Jacobi polynomials:

$$z = at^n P_{m-1}^{(n,-m)}(1-2t). \tag{5.10}$$

In particular we will need following asymptotic properties of this map:

$$z = a\frac{(m+n-1)!}{n!(m-1)!}t^n \quad \text{near} \quad z=0, \tag{5.11}$$

$$z = a + a\frac{(m+n-1)!}{m!(n-1)!}(t-1)^m \quad \text{near} \quad z=a, \tag{5.12}$$

$$z = a\frac{(m+n-2)!}{(m-1)!(n-1)!}t^{m+n-1} \quad \text{near} \quad z=\infty \tag{5.13}$$

The values of b_0 and b_1 can be seen from (5.11) and (5.12) to be:

$$b_0 = a\frac{(m+n-1)!}{n!(m-1)!} \qquad b_1 = a\frac{(m+n-1)!}{m!(n-1)!}. \tag{5.14}$$

Then the charge insertions at $t=0$ and $t=1$ give

$$b_0^{-p^2/n} : \exp\left(ipe_a\Phi^a(0)\right) : \qquad b_1^{-q^2/m} : \exp\left(iqe_a\Phi^a(1)\right) : \tag{5.15}$$

$$= n^{p^2/n} m^{q^2/m} \left(\frac{a(m+n-1)!}{(n-1)!(m-1)!}\right)^{-p^2/n-q^2/m} : \exp\left[ipe^b\Phi^b(0) + iqe^b\Phi^b(1)\right] : \tag{5.16}$$

To compute the contribution of (5.15) to the fusion coefficient we recall the way we had computed the fusion coefficients of the bosonic twists in [9]. The z plane was cut off at a large radius $|z| = 1/\delta$, and a second disc was glued in at this boundary to make the z plane into a sphere with an explicitly chosen metric. The map $z(t)$ taking multivalued functions on the z sphere to single valued functions on the covering space thus mapped a closed surface (a sphere) to a closed surface Σ (which in the present case is also a sphere). It was important to explicitly close all surfaces in order to use the argument of the conformal anomaly to compute the correlation function.

In the 3-point function we place operators at $z=0$, $z=a$ and $z=\infty$. We normalized the insertions at $z=0$ and $z=a$. We could also normalize the insertion at $z=\infty$ and directly compute the 3-point function, but because the operator at infinity is in a different coordinate patch it is easier to proceed in a slightly different way. We place an operator σ_q^∞ at infinity, without regard to its normalization, and compute the ratio

$$\frac{\langle \sigma_n(0)\sigma_m(a)\sigma_q^\infty(\infty)\rangle}{\langle \sigma_q(0)\sigma_q^\infty(\infty)\rangle} = C_{mnq} a^{-(\Delta_n+\Delta_m-\Delta_q)} \tag{5.17}$$

(where σ_q is correctly normalized) to compute the OPE coefficient.

To extend the same method to the present case we can get identical insertions at infinity in the case of $\sigma_n(0)\sigma_m(a)$ and the case of $\sigma_q(0)$ if we choose the map to the covering space for $\sigma_q(0)$ to agree at large z with the map to the covering space for the insertion $\sigma_n(0)\sigma_m(a)$. From (5.13) we see that the map for $\sigma_{m+n-1}(0)$ should be chosen to be

$$z = a\frac{(m+n-2)!}{(m-1)!(n-1)!} t^{m+n-1} \equiv b_\infty t^{m+n-1} \quad \text{near} \quad z = \infty \tag{5.18}$$

Writing $r = p+q$ for the charge of $\sigma^{\pm}_{n+m-1}(0)$ we get for the normalized charge insertion

$$|r\rangle = b_\infty^{-r^2/n} : \exp\left(ire_a\Phi^a(0)\right) :, \tag{5.19}$$

where

$$b_\infty = a\frac{(m+n-2)!}{(m-1)!(n-1)!}. \tag{5.20}$$

Then we get from the charge insertions the contribution to (3.18)

$$
\begin{aligned}
&\frac{\langle J,\mathcal{S}\rangle_{\sigma_1(0)\sigma_2(a)\sigma_3(\infty)}}{\langle J,\mathcal{S}\rangle_{\sigma_3(0)\sigma_3(\infty)}}\\
&=\frac{\langle b_0^{-p^2/n}:e^{ipe_a\Phi^a(0)}: \quad b_1^{-q^2/m}:e^{ipe_b\Phi^b(1)}: \quad :e^{-i(p+q)e_b\Phi^b(\infty)}:\rangle}{\langle b_\infty^{-r^2/(m+n-1)}:e^{i(p+q)e_c\Phi^c(0)}: \quad :e^{-i(p+q)e_d\Phi^d(\infty)}:\rangle}\\
&=b_0^{-p^2/n}b_1^{-q^2/m}b_\infty^{r^2/(m+n-1)}
\end{aligned}
\tag{5.21}
$$

Let us now evaluate $C^{---}_{n,m,n+m-1}$. In this case:

$$p=\frac{n-1}{2},\qquad q=\frac{m-1}{2},\qquad r=\frac{n+m-2}{2} \tag{5.22}$$

Combining the contribution (5.21) with the contribution (5.5) we get

$$|C^{---}_{n,m,n+m-1}|^2=|C_{n,m,n+m-1}|^{12}|a|^{-\frac{1}{2}(1-\frac{1}{n}-\frac{1}{m}+\frac{1}{n+m-1})}\ |b_0|^{-2p^2/n}|b_1|^{-2q^2/m}|b_\infty|^{2r^2/(m+n-1)} \tag{5.23}$$

where we now write the combined holomorphic and antiholomorphic sector contributions. Substituting the value of the bosonic fusion coefficient (5.5) and making algebraic simplifications, we get

$$\sigma_n^{--}(0)\sigma_m^{--}(a)\sim|C^{---}_{n,m,n+m-1}|^2\sigma^{--}_{n+m-1}(0) \tag{5.24}$$

with

$$|C^{---}_{n,m,n+m-1}|^2=\frac{m+n-1}{mn}. \tag{5.25}$$

Note that the power of a cancels out in (5.24), reflecting the fact that in the supersymmetric theory $\Delta^-_{m+n-1}=\Delta^-_n+\Delta^-_m$.

Similarly one finds

$$|C^{+-+}_{n,m,n+m-1}|^2=\frac{n}{m(m+n-1)} \tag{5.26}$$

6 General three point function for the sphere.

Let us now consider the general 3-point function $\langle\sigma_n^\pm\sigma_m^\pm\sigma_q^\pm\rangle$. The twist operators carry representations of the $su(2)\times su(2)$ symmetry group. Thus representations of the first two twist operators have to combine to give the (conjugate) to the representation of the third twist operator. Since there is at most one way to combine two representations of $su(2)$ to a third representation, we see that if we can compute the fusion coefficient for any chosen members of the three representations, then the general fusion coefficient can be deduced from this in terms of the Clebsch-Gordon coefficients.

In the simple case studied in the last section all operators had charge vectors of the form $|j,m\rangle=|Q,Q\rangle$ or $|j,m\rangle=|Q,-Q\rangle$, and so the charge could be represented as a

pure exponential. The main additional complication in the general case is that all charge vectors cannot be taken to have this simple form. We will let the operator at $z = 0$ have the form $|j, m\rangle = |Q, Q - d\rangle$, while the operator at $z = a$ has the form $|j, m\rangle = |P, P\rangle$ and the operator at $z = \infty$ has the form $|j, m\rangle = |Q - d + P, -(Q - d + P)\rangle$. Thus the operator at $z = 0$ needs to be modified by the action of lowering operators acting on an exponential, and we carry out this modification below. The other steps in the calculation parallel those of the previous section.

6.1 Constructing the state $|Q, Q - d\rangle$

We will construct a family of twist operators by acting on the highest weight state (4.27) by the charge operator corresponding to J^-. Using the fact that J_z^- has holomorphic dimension unity we can rewrite the corresponding charge on the covering space:

$$Q^- = \oint \frac{dz}{2\pi i} J^-(z) = \oint \frac{dz}{2\pi i} \sum_j \exp\left(-ie_a \phi_j^a(z)\right) = \oint \frac{dt}{2\pi i} \exp\left(-ie_a \Phi^a(t)\right). \tag{6.1}$$

Acting by this charge on the exponent (4.27), we get:

$$\oint \frac{dz}{2\pi i} J^-(z)|Q\rangle = b^{-Q^2/n} \oint_t \frac{dq}{2\pi i}(q - t)^{-2Q} : \exp\left(iQe_a\Phi^a(t) - ie_a\Phi^a(q)\right) : . \tag{6.2}$$

For the BPS operators constructed in section 4, the product $2Q$ is an integer, so one can evaluate the integral in (6.2):

$$Q^- : \exp\left(iQe_a\Phi^a(t)\right) := \frac{1}{(2Q-1)!}\partial_q^{2Q-1} : \exp\left(iQe_a\Phi^a(t) - ie_a\Phi^a(q)\right) : \Bigg|_{q=t} \tag{6.3}$$

Note that the action of Q^- does not change the conformal dimension of the operator, since we get a polynomial of order $2Q - 1$ multiplying the exponential, and $Q^2 = (Q - 1)^2 + (2Q - 1)$.

For the double action of the operator Q^- on BPS state one then gets:

$$\begin{aligned}
&(Q^-)^2 : \exp\left(iQe_a\Phi^a(t)\right) : \qquad (6.4)\\
&= \oint \frac{dq_2}{2\pi i} : \exp\left(-ie_a\Phi^a(q_2)\right) : \lim_{q_1\to t} \frac{1}{(2Q-1)!}\partial_{q_1}^{2Q-1} : \exp\left(iQe_a\Phi^a(t) - ie_a\Phi^a(q_1)\right) :\\
&= \lim_{q_1\to t} \frac{1}{(2Q-1)!}\partial_{q_1}^{2Q-1} \oint \frac{dq_2}{2\pi i}\frac{(q_2-q_1)^2}{(q_2-t)^{2Q}} : \exp\left(iQe_a\Phi^a(t) - ie_a\Phi^a(q_1) - ie_a\Phi^a(q_2)\right) :\\
&= \lim_{q_1\to t}\lim_{q_2\to t}\left(\frac{1}{(2Q-1)!}\partial_{q_1}^{2Q-1}\right)\left(\frac{1}{(2Q-1)!}\partial_{q_2}^{2Q-1}\right)\\
&\qquad \times (q_2-q_1)^2 : \exp\left(iQe_a\Phi^a(t) - ie_a\Phi^a(q_1) - ie_a\Phi^a(q_2)\right) :
\end{aligned}$$

One can generalize the above formula using induction:

$$
\begin{aligned}
(Q^-)^d\ :\exp\left(iQe_a\Phi^a(t)\right): \ &= \ \lim_{q_1\to t}\dots\lim_{q_d\to t}\prod_{l=1}^{d}\left(\frac{1}{(2Q-1)!}\partial_{q_l}^{2Q-1}\right) \\
&\times\prod_{i<j}^{d}(q_j-q_i)^2\ :\exp\left(iQe_a\Phi^a(t)-\sum_{l=1}^{d}ie_a\Phi^a(q_l)\right):
\end{aligned}
\tag{6.5}
$$

Let us evaluate the norm of the state (6.5). To do this we only need to use the structure of $SU(2)$ algebra:

$$
[Q^+,Q^-]=2Q^3,\qquad [Q^3,Q^\pm]=\pm Q^\pm. \tag{6.6}
$$

One can write the two point function of twist operators in terms of the norms of appropriate representations of $SU(2)$:

$$
\langle\left(\left(Q^-\right)^d\sigma_n^\pm(0)\right)\left(Q^+\right)^d\sigma_n^{\pm\dagger}(w)\rangle=\|\left(Q^-\right)^m|\frac{n\pm1}{2},\frac{n\pm1}{2}\rangle\|^2. \tag{6.7}
$$

Then standard manipulations give:

$$
\begin{aligned}
\langle Q,Q|\left(Q^+\right)^d\left(Q^-\right)^m|Q,Q\rangle &= 2\sum_{j=0}^{d-1}(Q-j)\,\langle Q,Q|\left(Q^+\right)^{d-1}\left(Q^-\right)^{d-1}|Q,Q\rangle \\
&= d(2Q+1-d)\|\left(Q^-\right)^{d-1}|Q,Q\rangle\|^2, \\
\|\left(Q^-\right)^d|Q,Q\rangle\| &= \left(\frac{d!(2Q)!}{(2Q-d)!}\right)^{1/2}\||Q,Q\rangle\|.
\end{aligned}
\tag{6.8}
$$

Thus we can define the normalized state:

$$
|Q,Q-d\rangle=\left(\frac{(2Q-d)!}{d!(2Q)!}\right)^{1/2}\oint\frac{dp_1}{2\pi i}J^-(p_1)\dots\oint\frac{dp_d}{2\pi i}J^-(p_d)|Q,Q\rangle. \tag{6.9}
$$

In the bosonised formulation we get:

$$
\begin{aligned}
|Q,Q-d\rangle &= \left(\frac{(2Q-d)!}{d!(2Q)!}\right)^{1/2}b^{-Q^2/n}\lim_{q_1\to t}\dots\lim_{q_d\to t}\prod_{l=1}^{d}\left(\frac{1}{(2Q-1)!}\partial_{q_l}^{2Q-1}\right) \\
&\times\prod_{i<j}^{d}(q_j-q_i)^2:\exp\left(iQe_a\Phi^a(t)-\sum_{l=1}^{d}ie_a\Phi^a(q_l)\right):
\end{aligned}
\tag{6.10}
$$

6.2 The contribution of the current insertions

Let us work out the contribution from the current insertions on the t sphere. This contribution corresponds to the factor in (3.18)

$$
\frac{\langle J,\mathcal{S}\rangle_{\sigma_1(0)\sigma_2(a)\sigma_3(\infty)}}{\langle J,\mathcal{S}\rangle_{\sigma_3(0)\sigma_3(\infty)}} \tag{6.11}
$$

We are looking for the fusion coefficient

$$(Q, Q - d_2) \times (P, P) \to (Q + P - d, Q + P - d), \tag{6.12}$$

one should consider the fusion rule for the following CFT operators:

$$|Q, Q - d\rangle_0 \ |P, P\rangle_a \approx \mathcal{A}|Q + P - d, Q + P - d\rangle_0. \tag{6.13}$$

We first note that

$$\begin{aligned}
&\Big\{(Q^-)^d : \exp\left(iQe_a\Phi^a(0)\right) :\Big\} : \exp\left(iPe_a\Phi^a(1)\right) : & (6.14)\\
&\quad \approx \lim_{q_i\to 0} \prod_{l=1}^{d} \left(\frac{1}{(2Q-1)!}\partial_{q_l}^{2Q-1}\right) \prod_{i<j}^{d} (q_j - q_i)^2 \prod_{k=1}^{d} (1 - q_k)^{-2P} \\
&\quad \times : \exp\left(i(P + Q - d)e_a\Phi^a(0)\right)[1 + \ldots] : & (6.15)
\end{aligned}$$

where the ... in the last term indicate polynomials in $\partial\Phi_a(0), \partial^2\Phi_a(0)$ etc. We are again using a computation analogous to (5.17), and will thus be computing the correlator of the above expression with a pure exponential operator at infinity. In this situation the terms represented by ... in the above expression will give a contribution that is vanishing compared to the first term, and can thus be ignored. Taking into account the normalization (6.10), we finally get:

$$\begin{aligned}
\mathcal{A} &= b_0^{-Q^2/n} b_1^{-P^2/n} b_\infty^{S^2/q} \left(\frac{(2Q-d)!}{d!(2Q)!}\right)^{1/2} \\
&\times \lim_{q_i\to 0} \prod_{l=1}^{d} \left(\frac{1}{(2Q-1)!}\partial_{q_l}^{2Q-1}\right) \prod_{i<j}^{d} (q_j - q_i)^2 \prod_{k=1}^{d} (1 - q_k)^{-2P}
\end{aligned} \tag{6.16}$$

The value of S in the above expression is $S = \frac{q\pm 1}{2}$ for $\sigma_q^\pm$.

Notation: It will be helpful in what follows to introduce the following notation. We will characterize the twist operator $\sigma_n^\pm$ by two numbers: n and 1_n, where $1_n = 1$ for σ_n^+ and $1_n = -1$ for σ_n^-. We also introduce the notation

$$\sigma_n^{1_n(d)} \tag{6.17}$$

for twist operator corresponding to the state (6.10). Here the value of Q is given by $Q = \frac{n+1_n}{2}$ and d is the number of lowering operators Q^- that have been applied to the state $|Q, Q\rangle$.

To evaluate the expression $\mathcal{A}$ we need to use the map which takes the z sphere to the t sphere. In [9] we have shown that such a map is unique up to $SL(2, C)$ transformation

and its explicit form is given in the appendix B. Here we will need only some properties of this map, namely its behavior near the ramification points:

$$t \to 0: \qquad z \approx a \frac{d_1! d_2! (n - d_2 - 1)!}{n!(d_1 - n)!(n - 1)!} t^n, \tag{6.18}$$

$$t \to 1: \qquad z \approx a + a \frac{d_1! d_2! (m - d_2 - 1)!}{m!(d_1 - m)!(m - 1)!} (t - 1)^m, \tag{6.19}$$

$$t \to \infty: \qquad z \approx a \frac{d_2! (d_1 - d_2 - 1)! (d - 1 - d_2)!}{d_1! (d_1 - n)! (d_1 - m)!} t^{d_1 - d_2}. \tag{6.20}$$

Using the above information, we get:

$$\begin{aligned}
\log \mathcal{A} &= \left(-\frac{(n + 1_n)^2}{4n} - \frac{(m + 1_m)^2}{4m} - \frac{(q + 1_q)^2}{4q} \right) \log d_1! \\
&+ \left\{ \left(\frac{(n{+}1_n)^2}{4n} - \frac{(m{+}1_m)^2}{4m} - \frac{(q{+}1_q)^2}{4q} \right) \log(d_1{-}n)! + (n \leftrightarrow m) + (n \leftrightarrow q) \right\} \\
&+ \left\{ \frac{(n + 1_n)^2}{4n} \log \left(n!(n - 1)! \right) + (n \leftrightarrow m) + (n \leftrightarrow q) \right\} \\
&+ \log \left\{ \lim_{q_i \to 0} \prod_{l=1}^{d} \left(\frac{1}{(n + 1_n - 1)!} \partial_{q_l}^{n + 1_n - 1} \right) \prod_{i<j}^{d} (q_j - q_i)^2 \prod_{k=1}^{d} (1 - q_k)^{-m - 1_m} \right\} \\
&- \frac{1}{2} \log \left(\frac{d!(n + 1_n)!}{(n + 1_n - d)!} \right) + \left(\frac{S^2}{q} - \frac{P^2}{n} - \frac{Q^2}{m} \right) \log a,
\end{aligned} \tag{6.21}$$

where

$$d = d_2 + \frac{1}{2}(1_n + 1_m - 1_q + 1). \tag{6.22}$$

6.3 The fusion coefficient $C_{n,m,q}^{---}$

Collecting the contributions from the conformal anomaly and charge insertions, one finds the following OPE

$$\sigma_n^{1_n(d)}(0) \sigma_m^{1_m}(a) \quad \approx \quad a^{\Delta_q^{1_q} - \Delta_n^{1_n} - \Delta_m^{1_m}} C_{n,m,q}^{1_n(d), 1_m, 1_q} \sigma_q^{1_q}(0), \tag{6.23}$$

$$C_{n,m,q}^{1_n(d), 1_m, 1_q} \quad = \quad (C_{n,m,q})^6 \mathcal{A} a^{-S^2/q + P^2/n + Q^2/m} \tag{6.24}$$

Here we are writing only the holomorphic part of the OPE; the complete OPE will have representations of the two $su(2)$ factors and will be constructed at the end.

Taking into account the expression for fusion coefficient of the bosonic twists found in [9] (see appendix B), we finally get an expression for the holomorphic part of the fusion

coefficient

$$\log C^{1_n(d),1_m,1_q}_{n,m,q} = (-d_2+2)\log d_1! + d_2\left(\log(d_1-n)! + \log(d_1-m)!\right) \tag{6.25}$$

$$+(d_2-2)\log d_2! - 2\log(q-1)! - \frac{1}{2}\log(mnq^3) + d_2(d_2-1)\log 2 + \log\mathcal{D}$$
$$+\log L^d_{n+1_n,m+1_m} - \frac{1}{2}\log\left(\frac{d!(n+1_n)!}{(n+1_n-d)!}\right) - \frac{1}{2}(3+1_n+1_m+1_q)\log d_1! +$$
$$\left\{\frac{1_n-1_m-1_q-1}{2}\log(d_1-n)! + \frac{1+1_n}{2}\log\left(n!(n-1)!\right) + (n\leftrightarrow m) + (n\leftrightarrow q)\right\}.$$

Here we introduced the notation:

$$L^d_{n,m} = \left\{\lim_{q_i\to 0}\prod_{l=1}^{d}\left(\frac{1}{(n-1)!}\partial^{n-1}_{q_l}\right)\prod_{i<j}^{d}(q_j-q_i)^2\prod_{k=1}^{d}(1-q_k)^{-m}\right\}. \tag{6.26}$$

To get a contribution form antiholomorphic sector, one should replace 1_n, 1_m and 1_q in (6.25) by corresponding value for antiholomorphic part ($\bar{1}_n$, $\bar{1}_m$ or $\bar{1}_q$).

The expression (6.25) looks complicated for two reasons. Firstly it contains the discriminant $\mathcal{D}$ which is known only as a finite product (B.6). Secondly there is the term $L^d_{n,m}$ defined through derivatives and limits.

First we note that (6.25) simplifies enormously for the low values of d_2. In particular, one can easily evaluate $C^{---}_{n,m,q}$ for $d_2=0$ and $d_2=1$:

$$C^{-,-,-}_{n,m,m+n-1} = \left(\frac{m+n-1}{mn}\right)^{1/2}, \tag{6.27}$$

$$C^{-(1),-,-}_{n,m,m+n-3} = \left(\frac{(m+n-2)^2}{mn(m+n-3)}\frac{(n-2)!}{(n-1)!}\right)^{1/2}. \tag{6.28}$$

The first of these two expressions is the result (5.26) derived for a special case in the previous section. Investigating a few further cases, and using the symmetry properties of C lead us to guess that the complicated expression (6.25) for the case of three operators σ^- is equal to the following simple expression:

$$C^{-(d_2),-,-}_{n,m,q} = \left(\frac{d_1^2(d_2!)^2}{mnq}\frac{(n-d_2-1)!}{d_2!(n-1)!}\right)^{1/2}. \tag{6.29}$$

While we could not prove the agreement of (6.25) and (6.29) analytically, we have verified it for $d_2\le 5$ and arbitrary values of n and m using a symbolic manipulations program (Mathematica). Assuming the equality of these two forms of the result, one arrives at the following expression for the $L^d_{n,m}$:

$$L^d_{n,m} = (d!)^3\left[\frac{(n+m+1-d)!}{d!(n-d)!(m-d)!}\right]^d\left[\frac{(n+m+1-2d)!}{(n+m+1-d)!}\right]^2\frac{n+m+1-d}{n+m+1-2d} \tag{6.30}$$

$$\times\prod_{j=1}^{d} j^{2d-2-j}(j-n-1)^{1-j}(j-m-1)^{1-j}(j-n-m-2+d)^{j-d} \tag{6.31}$$

6.4 The fusion coefficients for general elements of the representations

From now on we will assume that (6.31) is correct, and use this to write compact expressions for general fusion coefficients.

We can rewrite (6.29) in the more symmetric form:

$$C^{-(d_2),-,-}_{n,m,q} = \left(\frac{d_1^2}{mnq}\frac{[(n+m-q-1)/2]![(n-m+q-1)/2]!}{(n-1)!}\right)^{1/2} \tag{6.32}$$

Since we are considering the fusion of the different representations of $SU(2)$, we anticipate the appearance of the appropriate $3j$ symbols in the final answer. In terms of the standard notation $|j,m\rangle$ for the representations of $SU(2)$, the operator $\sigma^-_n(0)$ can be written as

$$|\frac{n-1}{2}, \frac{n-1}{2} - d_2\rangle, \tag{6.33}$$

so the case computed in the above subsection corresponds to the following $3j$ symbol:

$$\begin{pmatrix} \frac{n-1}{2} & \frac{m-1}{2} & \frac{q-1}{2} \\ \frac{q-m}{2} & \frac{m-1}{2} & \frac{1-q}{2} \end{pmatrix}. \tag{6.34}$$

One can easily evaluate this particular type of the $3j$ symbol (see, for example [18]):

$$\begin{pmatrix} \frac{n-1}{2} & \frac{m-1}{2} & \frac{q-1}{2} \\ \frac{q-m}{2} & \frac{m-1}{2} & \frac{1-q}{2} \end{pmatrix} = \left(\frac{(m-1)!(q-1)!}{d_1![(m+q-n-1)/2]!}\right)^{1/2}. \tag{6.35}$$

We are to take the positive sign of the square root here and in all similar expressions in what follows. The fact that the coefficients (6.35) are positive allow us to take such square roots and thus write separate factors for the holomorphic and aniholomorphic fusion coefficients, without introducing any extra phase factors.

Using the above expression, we can rewrite (6.32) in the final form:

$$\begin{aligned} C^{-(d_2),-,-}_{n,m,q} &= \left(\frac{d_1^2}{mnq}\frac{d_1!\ [\frac{n+m-q-1}{2}]!\ [\frac{n+q-m-1}{2}]!\ [\frac{m+q-n-1}{2}]!}{(n-1)!\ (m-1)!\ (q-1)!}\right)^{1/2} \\ &\times \begin{pmatrix} \frac{n-1}{2} & \frac{m-1}{2} & \frac{q-1}{2} \\ \frac{q-m}{2} & \frac{m-1}{2} & \frac{1-q}{2} \end{pmatrix}. \end{aligned} \tag{6.36}$$

In particular, it is convenient to define the reduced fusion coefficient, which describes a fusion of different representations of $SU(2)$ and does not specify the "orientation of the spins".

$$\hat{C}^{---}_{n,m,q} = \left(\frac{d_1^2}{mnq}\frac{d_1!\ [\frac{n+m-q-1}{2}]!\ [\frac{n+q-m-1}{2}]!\ [\frac{m+q-n-1}{2}]!}{(n-1)!\ (m-1)!\ (q-1)!}\right)^{1/2} \tag{6.37}$$

Then the fusion rule for the specific example of twist operators σ^{--} reads:

$$\begin{aligned} &\sigma_n^{--(s_1,\bar{s}_1)}(0)\ \sigma_m^{--(s_2,\bar{s}_2)}(a) \sim |a|^{-(2\Delta_n+2\Delta_m-2\Delta_q)}\sigma_q^{--(s_3,\bar{s}_3)}(0) \\ &\qquad \times|\hat{C}^{---}_{n,m,q}|^2 \begin{pmatrix} \frac{n-1}{2} & \frac{m-1}{2} & \frac{q-1}{2} \\ s_1 & s_2 & s_3 \end{pmatrix} \begin{pmatrix} \frac{n-1}{2} & \frac{m-1}{2} & \frac{q-1}{2} \\ \bar{s}_1 & \bar{s}_2 & \bar{s}_3 \end{pmatrix} \end{aligned} \tag{6.38}$$

6.5 The fusion coefficients $C_{m,n,q}^{1_n,1_m,1_q}$

We have also analyzed the general fusion coefficients $C_{m,n,q}^{1_n,1_m,1_q}$. We present the computation of another case, the coefficient $C_{m,n,q}^{+-+}$, in appendix C. The final result for the reduced fusion coefficients is:

$$\hat{C}_{n,m,q}^{1_n,1_m,1_q} = \left(\frac{(1_n n + 1_m m + 1_q q + 1)^2}{4mnq} \frac{\Sigma!\ \alpha_n!\alpha_m!\alpha_q!}{(n+1_n)!\ (m+1_m)!\ (q+1_q)!} \right)^{1/2} \tag{6.39}$$

where

$$\Sigma = \frac{1}{2}(n + 1_n + m + 1_m + q + 1_q) + 1, \tag{6.40}$$

$$\alpha_n = \Sigma - n - 1_n - 1 \tag{6.41}$$

Note that the parameters $1_n, 1_m, 1_q$ can be chosen independently for the holomorphic and antihlomorphic parts of the twist operator. The full fusion coefficient is then a product of (6.39) from the left and right sides, together with the Clebsch-Gordon coefficients from the left and right $su(2)$ representations.

6.6 Combinatoric factors and large N limit.

The twist operators we have considered so far do not represent proper fields in the conformal field theory. In the orbifold CFT there is one twist field for each conjugacy class of the permutation group, not for each element of the group [8]. The true CFT operators that represent the twist fields can be constructed by summing over the group orbit, for example:

$$O_n^{1_n\bar{1}_n} = \frac{\lambda_n}{N!} \sum_{h\in G} \sigma_{h(1\ldots n)h^{-1}}^{1_n\bar{1}_n} \ . \tag{6.42}$$

Here G is the permutation group S_N and the normalization constant λ_n can be determined from the normalization condition. Namely if one starts from normalized σ operators:

$$\langle \sigma_n^{1_n\bar{1}_n}(0)\sigma_n^{1_n\bar{1}_n\dagger}(1)\rangle = 1, \tag{6.43}$$

then for O we get:

$$\langle O_n^{1_n\bar{1}_n}(0)O_n^{1_n\bar{1}_n\dagger}(1)\rangle = \lambda_n^2 n \frac{(N-n)!}{N!} \langle \sigma_n^{1_n\bar{1}_n}(0)\sigma_n^{1_n\bar{1}_n\dagger}(1)\rangle. \tag{6.44}$$

Requiring the normalization $\langle O_n^{1_n\bar{1}_n}(0)O_n^{1_n\bar{1}_n\dagger}(1)\rangle = 1$, we find the value of λ_n:

$$\lambda_n = \left[\frac{n(N-n)!}{N!}\right]^{-1/2} . \tag{6.45}$$

Let us now look at the three point function. For the case when the covering surface is a sphere, the three point function is [9]:

$$\langle O_n^{1_n\bar{1}_n}(0)O_m^{1_m\bar{1}_m}(1)O_q^{1_q\bar{1}_q\dagger}(z)\rangle = \frac{\sqrt{mnq(N-n)!(N-m)!(N-q)!}}{(N-s)!\sqrt{N!}}$$
$$\times\langle\sigma_n^{1_n\bar{1}_n}(0)\sigma_m^{1_m\bar{1}_m}(1)\sigma_q^{1_q\bar{1}_q\dagger}(z)\rangle.$$

with $s = \frac{1}{2}(n+m+q-1)$.

Now we analyze the behavior of the combinatoric factors for arbitrary genus g but in the limit where N is taken to be large while the orders of twist operators (m, n and q) as well as the parameter g are kept fixed. There are s different fields X^i involved in the 3-point function, and these fields can be selected in $\sim N^s$ ways. Similarly the 2-point function of σ_n will go as N^n since n different fields are to be selected. Thus the 3-point function of normalized twist operators will behave as

$$N^{s-\frac{n+m+q}{2}} = N^{-(g+\frac{1}{2})} \tag{6.46}$$

Thus in the large N limit the contributions from surfaces with high genus will be suppressed, and the leading order the answer can be obtained by considering only contributions from the sphere ($g = 0$). This is presicely the case that we have analyzed in detail, and knowing the amplitude for operators σ one can easily extract the leading order of the CFT correlation function:

$$\langle O_n^{1_n\bar{1}_n}(0)O_m^{1_m\bar{1}_m}(1)O_q^{1_q\bar{1}_q}(z)\rangle = \sqrt{\frac{1}{N}}\sqrt{mnq}\langle\sigma_n^{1_n\bar{1}_n}(0)\sigma_m^{1_m\bar{1}_m}(1)\sigma_q^{1_q\bar{1}_q}(z)\rangle_{sphere} + O\left(\frac{1}{N^{3/2}}\right). \tag{6.47}$$

7 Discussion

In this paper we have computed the contribution to the 3-point function for the case when the covering surface Σ is a sphere. The construction of the chiral operators and the method of computing correlators used only the fact that the CFT based on the manifold M had $N = 4$ supersymmetry; thus the computation is not restricted to orbifolds that can be obtained from free fields. The result we obtain is independent of the details of M, and thus exhibits a 'universal' property of CFTs arising from orbifolds M^N/S^N.

The sphere contribution is the dominant contribution at large N, but we can in principle calculate the contribution of surfaces Σ with $g > 0$ to get an exact result for finite N. Note that for a given choice of orders of the chiral operators, the number of different surfaces Σ that can contribute is finite, though it grows with the orders of the operators. In the case of the bosonic orbifold we had shown in [9] that the correlator for twist operators on the z plane for the orbifold M^N/S^N could be written in terms of the partition functions Z_g for the theory with one copy of M but on a worldsheet of genus g.

For the supersymmetric case we get the following extension of this result: the correlator of the chiral operators considered here can be written in terms of the Z_g and a finite number of derivatives of Z_g with respect to moduli, where the moduli arise from both the shape of the Riemann surface and from the 'current algebra moduli' that couple to the $su(2)$ currents. This result follows from the fact that we map the correlator of twist operators to the covering surface Σ as in the bosonic case, but we get some correlators of currents that need to be computed on Σ. The current insertions can be removed by using the current algebra Ward identity, but in the process we generate derivatives with respect to the moduli.

The supersymmetric orbifold that we have studied is expected to be a point in the D1-D5 system moduli space, and the latter system is dual to string theory on $AdS_3 \times S^3 \times M$. In this context we recall some observations that were made in [9] relating orbifold correlation functions to the dual string theory. First, it was found that to get a leading order 3-point function of twist operators (which requires that the genus of Σ be zero) we must satisfy some restrictions on the orders of the twists. These restrictions turn out to be identical to the fusion rules of the $su(2)$ WZW model, which restrict the 3-point functions of string states at tree level in the dual string theory. Secondly, we note that the orbifold theory has expressed the correlators of twist operators as a sum of contributions from different genus Riemann surfaces, with the contribution of higher genera surfaces being suppressed by $1/N^g$. This is reminiscent of the genus expansion in the dual string theory, where higher genus amplitudes are suppressed by a similar factor ($1/\sqrt{N}$ is the coupling constant of the string modes).

It would be interesting to compare the 3-point functions that we have computed to the 3-point functions of supergravity field in the dual theory, to see if there is any analogue of the nonrenormalization that was found in the case of D-3 branes and the dual theory on $AdS_5 \times S^5$. Some 3-point supergravity amplitudes have been computed in [21], but these correspond to scalar fields which are expected to be supersymmetry descendents of the chiral fields that we have worked with. Another supergravity 3-point correlation function was computed in [14], and there is a significant agreement in overall form between this result and the correlators that we get. However the calculation of [14] used a step where a symmetrization was performed over the three fields involved in the correlator, and to reproduce the result obtained from the orbifold computation we would have to choose a somewhat different way to symmetrize (instead of summing the squares of the three momenta we would have to sum the momenta and then square the result). We hope to return to this comparison at a later point.

The D1-D5 system for the black holes studied in [1][2] arises by wrapping the space direction of the D1-D5 CFT on a circle, with periodic boundary conditions for the fermions on this circle. Thus we need to study the CFT in the Ramond sector, rather than in the NS sector. We can compute correlation functions in the Ramond sector if we can compute correlators with insertions of spin fields, since these spin fields map the vacuum on the plane to the Ramond vacuum. (These insertions of spin fields are different from the spin fields that we have encountered in the present paper; the latter arose only after

the map to the t space, while the former would be inserted to change boundary conditions in the original z space.) We have computed examples of correlators in the Ramond sector, which correspond to the amplitude for a simple 'black hole state' to absorb and re–emit a quantum; this calculation will be presented elsewhere [22]. In the dual theory on $AdS_3 \times S^3 \times M$ the NS and Ramond sector are connected by a spectral flow as well. It was suggested in [23] that a family of singular conical defect spacetimes would represent this spectral flow, but it was argued recently in [24] that the generic spacetime in the flow would in fact be smooth, and the spacetime deformation corresponding to spectral flow away from the NS vacuum was computed to first order .

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A $N=4$ superconformal algebra.

$$\begin{aligned}
T(z)T(w) &= \frac{\partial T(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \frac{c}{2(z-w)^4}, \\
J^i(z)J^j(w) &= \frac{i\varepsilon^{ijk}J^k(w)}{z-w} + \frac{c}{12(z-w)^2}, \\
T(z)J^i(w) &= \frac{\partial J^i(w)}{z-w} + \frac{J^i(w)}{(z-w)^2}, \\
G^a(z)\tilde{G}_b(w) &= \frac{2T(w)\delta^a_b}{z-w} - \frac{2(\sigma^i)^a{}_b\partial J^i(w)}{z-w} - \frac{4(\sigma^i)^a{}_b J^i(w)}{(z-w)^2} + \frac{2c\delta^a_b}{3(z-w)^3}, \qquad (A.1) \\
T(z)G^a(w) &= \frac{\partial G^a(w)}{z-w} + \frac{3G^a(w)}{2(z-w)^2}, \qquad T(z)\tilde{G}_a(w) = \frac{\partial \tilde{G}_a(w)}{z-w} + \frac{3\tilde{G}_a(w)}{2(z-w)^2}, \\
J^i(z)G^a(w) &= -\frac{(\sigma^i)^a{}_b G^b(w)}{2(z-w)}, \qquad J^i(z)\tilde{G}_a(w) = \frac{\tilde{G}_b(w)(\sigma^i)^b{}_a}{2(z-w)}
\end{aligned}$$

The elements of G^a and $\tilde{G}_b$ are related by complex conjugation:

$$\tilde{G}_a(z) = (G^a(z))^\dagger\,. \qquad (A.2)$$

In terms of modes we have:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

$$
\begin{aligned}
\left[J^i_m, J^j_n\right] &= i\varepsilon^{ijk}J^k_{m+n} + \frac{c}{12}m\delta_{m+n,0}, \qquad \left[L_m, J^i_n\right] = -nJ^i_{m+n}, \\
\left\{G^a_r, \tilde{G}_{b,s}\right\} &= 2\delta^a_b L_{r+s} - 2(r-s)(\sigma^i)^a{}_b J^i_{r+s} + \frac{c(4r^2-1)}{12}\delta^a_b\delta_{r+s,0}, \\
\left\{G^a_r, G^b_s\right\} &= 0, \qquad \left\{\tilde{G}_{a,r}, \tilde{G}_{b,s}\right\} = 0 \\
\left[L_m, G^a_r\right] &= \left(\frac{m}{2} - r\right)G^a_{m+r}, \qquad \left[L_m, \tilde{G}_{a,r}\right] = \left(\frac{m}{2} - r\right)\tilde{G}_{a,m+r} \\
\left[J^i_m, G^a_r\right] &= -\frac{1}{2}(\sigma^i)^a{}_b G^b_{m+r}, \qquad \left[J^i_m, \tilde{G}_{a,r}\right] = \frac{1}{2}\tilde{G}_{b,m+r}(\sigma^i)^b{}_a
\end{aligned} \tag{A.3}
$$

As was pointed out by Schwimmer and Seiberg [19], there is a family of equivalent representations of the superconformal algebra, which differ only by the boundary conditions:

$$
J^{\pm}(z) = e^{\mp 2i\pi\eta}J^{\pm}(e^{2i\pi}z), \qquad G^1(z) = -e^{i\pi\eta}G^1(e^{2i\pi}z), \qquad \tilde{G}_2(z) = -e^{i\pi\eta}\tilde{G}_2(e^{2i\pi}z). \tag{A.4}
$$

The equivalence can be established by the action of the spectral flow [3]:

$$
T_\eta(z) = T(z) - \frac{\eta}{z}J^3(z) + \frac{c\eta^2}{24z^2}, \tag{A.5}
$$

$$
J^3_\eta(z) = J^3(z) - \frac{c\eta}{12z}, \qquad J^{\pm}_\eta(z) = z^{\mp\eta}J^{\pm}(z), \tag{A.6}
$$

$$
G^1_\eta(z) = z^{\frac{\eta}{2}}G^1(z), \qquad G^2_\eta(z) = z^{-\frac{\eta}{2}}G^2(z), \tag{A.7}
$$

$$
\tilde{G}_{1\eta}(z) = z^{-\frac{\eta}{2}}\tilde{G}_1(e^{2i\pi}z), \qquad \tilde{G}_{2\eta}(z) = z^{\frac{\eta}{2}}\tilde{G}_2(e^{2i\pi}z) \tag{A.8}
$$

In particular, if we started with "vacuum state" $|\chi\rangle$:

$$
\oint \frac{dz}{2\pi i}z^{p+1}T(z)|\chi\rangle = \oint \frac{dz}{2\pi i}z^p J^3(z)|\chi\rangle = 0, \qquad p \geq 0. \tag{A.9}
$$

in $\eta = 0$ sector, then the charges in η sector are given by:

$$
\oint \frac{dz}{2\pi i}zT_\eta(z)|\chi\rangle = \frac{c\eta^2}{24}, \qquad \oint \frac{dz}{2\pi i}J^3_\eta(z)|\chi\rangle = -\frac{c\eta}{12}, \qquad c = 6. \tag{A.10}
$$

To have a well defined expression for $J(z)$ one has to choose an odd integer η, but for odd η the fermionic operators become integer–moded, i.e. we obtain a Ramond sector of the theory. We will pick the simplest possible value: $\eta = 1$. Thus, starting point of our flow (i.e. NS sector) corresponds to $\eta = 0$, while at $\eta = 1$ we get a Ramond sector. We will also need relations between modes in NS and R sectors (primed modes are from the R sector):

$$
\begin{aligned}
T'_n = T_n - J^3_n + \frac{c}{24}\delta_{n,0}, &\qquad (J^3)'_n = J^3_n - \frac{c}{12}\delta_{n,0}, \qquad (J^{\pm})'_n = J^{\pm}_{n\mp 1}, \\
(G^1)'_n = G^1_{n+\frac{1}{2}}, \qquad (G^2)'_n = G^2_{n-\frac{1}{2}}, &\qquad \tilde{G}'_{1,n} = \tilde{G}_{1,n-\frac{1}{2}}, \qquad \tilde{G}'_{2,n} = \tilde{G}'_{2,n+\frac{1}{2}}
\end{aligned} \tag{A.11}
$$

[3]Since the R symmetry group for $N = 4$ theory is $SU(2)$, there is a three parametric family of spectral flows. We will perform such flow only along J^3 direction

In the NS sector there was a bound on a maximal possible charge: $|j| \leq h$, which saturated only for (anti)chiral fields. In particular, chiral fields had $h = j$. In the Ramond sector this bound becomes: $h' \geq \frac{c}{24}$ and it is saturated only by images of chiral fields.

B Three point function for the bosonic orbifold.

In this appendix we will summarize the results of [9], where the three point function on the bosonic S^N orbifold was calculated. In particular, in [9] we evaluated the contribution to the three point function for the case where the covering space is a sphere. In this case the map corresponding to the three point function

$$\langle \sigma_n(0)\sigma_m(a)\sigma_q(\infty)\rangle \tag{B.1}$$

is given by:

$$z = at^n P_{d_1-n}^{(n,-d_1-d_2+n-1)}(1-2t)\left[P_{d_2}^{(-n,-d_1-d_2+n-1)}(1-2t)\right]^{-1}. \tag{B.2}$$

This map involves Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and the values of d_1 and d_2 are defined as

$$d_1 = \frac{1}{2}(n+m+q-1), \qquad d_2 = \frac{1}{2}(n+m-q-1). \tag{B.3}$$

One can easily show that the asymptotic behavior of this map is given by (6.18), (6.19), (6.20). We recall that the three point function (B.1) was evaluated in [9] by considering the conformal anomaly for the map (B.2). More precisely, this three point function can be written in terms of the Liouville action describing the map (B.2) and the Liouville actions corresponding to two point functions $\langle \sigma_n(0)\sigma_n(z)\rangle$:

$$\begin{aligned}\frac{\langle \sigma_n(0)\sigma_m(a)\sigma_q(\infty)\rangle}{\langle \sigma_q(0)\sigma_q(\infty)\rangle} &= \exp\left(S_L(\sigma_n(0),\sigma_m(a),\sigma_q(\infty)) - S_L(\sigma_q(0)\sigma_q(\infty))\right)\\ &\times \exp\left(-\frac{1}{2}S_L(\sigma_n(0),\sigma_n(1)) - \frac{1}{2}S_L(\sigma_m(0),\sigma_m(1)) + \frac{1}{2}S_L(\sigma_q(0),\sigma_q(1))\right)\\ &\equiv |C_{n,m,q}|^{2c}|a|^{-\frac{c}{12}(n+m-q-1/n-1/m+1/q)}\end{aligned} \tag{B.4}$$

$C_{n,m,q}$ is the fusion coefficient of σ_n and σ_m to σ_q.

The evaluation of the three point function leads to the following result [9]:

$$\begin{aligned}\log|C_{n,m,q}|^2 &= \frac{1}{6}\log\left(\frac{q}{mn}\right) - \frac{n-1}{12}\log n - \frac{m-1}{12}\log m + \frac{q-1}{12}\log(q)\\ &- \frac{n-1}{12n}\log\left(\frac{d_1!d_2!}{n!(n-1)!}\,\frac{(d_1-m)!}{(d_1-n)!}\right)\\ &- \frac{m-1}{12m}\log\left(\frac{d_1!d_2!}{m!(m-1)!}\,\frac{(d_1-n)!}{(d_1-m)!}\right)\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{q-1}{12q}\log\left(\frac{(q-1)!d_2!}{(d_1-n)!(d_1-m)!}\ \frac{(d_1-d_2)!)}{d_1!}\right) \\
&+ \quad \frac{1}{3}d_2(d_2-1)\log 2 - \frac{d_2}{6}\log n + \frac{1}{3}\log\mathcal{D} + \frac{3d_2-4}{6}\log d_2! \\
&- \quad \frac{d_2}{6}\log\left[\frac{d_1!}{n!(d_1-n)!}\right] - \frac{n+d_2-1}{6}\log\frac{(n-1)!}{(n-d_2-1)!} \\
&- \quad \frac{d_1-d_2+3}{6}\log\frac{(d_1-d_2)!}{d_1!} - \frac{d_1+d_2-n}{6}\log\frac{(d_1+d_2-n)!}{(d_1-n)!}.
\end{aligned}
\tag{B.5}
$$

This expression involves the discriminant of the Jacobi polynomials $\mathcal{D}$, which is known as a following finite product [20]:

$$
\begin{aligned}
\mathcal{D} \quad &\equiv \quad D_{d_2}^{(-n,-d_1-d_2+n-1)} = 2^{-d_2(d_2-1)} \\
\times \quad &\prod_{j=1}^{d_2} j^{j+2-2d_2}(j-n)^{j-1}(j-d_1-d_2+n-1)^{j-1}(j-d_1-1)^{d_2-j}.
\end{aligned}
\tag{B.6}
$$

Note that the combination of Liouville actions entering (B.4) is the combination needed for SUSY orbifold (3.13). Thus we can take the result for $|C_{n,m,q}|$ as contribution of the conformal anomaly even for supersymmetric case.

C Calculation of $\hat{C}^{+,-,+}_{n,m,q}$.

In section 6 we have calculated $\hat{C}^{-,-,-}_{n,m,q}$ and presented the result for general reduced fusion coefficient $\hat{C}^{1_n,1_m,1_q}_{n,m,q}$. In this section we will assume the relation (6.26) and, using it, we will evaluate another fusion coefficient $\hat{C}^{+,-,+}_{n,m,q}$ which will give another case of the result (6.39). Using the general expression (6.25), one can find a ratio:

$$
\frac{C^{+(d_2),-,+}_{n,m,q}}{C^{-(d_2),-,-}_{n,m,q}} = \frac{n!(n-1)!q!(q-1)!}{(d_1!(d_1-m)!)^2}\left(\frac{(n-d_2)(n-d_2+1)}{n(n+1)}\right)^{1/2}\frac{L^{d_2}_{n+1,m-1}}{L^{d_2}_{n-1,m-1}}. \tag{C.1}
$$

To evaluate a ratio of limits one can use the fact that for an integer $l>1$:

$$
\prod_{j=1}^{d}\left[\frac{j-l-2}{j-l}\right]^{1-j} = \left(\frac{(l-d-1)!}{(l-1)!}\right)^2\frac{(l-d)^{d+1}(l-d+1)^d}{l}. \tag{C.2}
$$

Then one gets:

$$
\frac{L^{d_2}_{n+1,m-1}}{L^{d_2}_{n-1,m-1}} = \left[\frac{(n-d_2-1)!(m+n-d_2-1)!}{(m+n-2d_2-1)!(n-1)!}\right]^2\frac{n-d}{n}\frac{m+n-2d_2-1}{m+n-d_2-1} \tag{C.3}
$$

To define the reduced fusion coefficient we also need the relation between appropriate $3j$ symbols:

$$
\begin{pmatrix} \frac{n+1}{2} & \frac{m-1}{2} & \frac{q+1}{2} \\ \frac{q-m+2}{2} & \frac{m-1}{2} & -\frac{1+q}{2} \end{pmatrix} = \left[\frac{q(q+1)}{(d_1+1)(d_1+2)}\right]^{1/2}\begin{pmatrix} \frac{n-1}{2} & \frac{m-1}{2} & \frac{q-1}{2} \\ \frac{q-m}{2} & \frac{m-1}{2} & \frac{1-q}{2} \end{pmatrix}, \tag{C.4}
$$

which can be deduced from the general expression:

$$\begin{pmatrix} a & b & c \\ c-b & b & -c \end{pmatrix} = \left(\frac{[2b]![2c]!}{[a+b+c+1]![b+c-a]!}\right)^{1/2}. \tag{C.5}$$

Collecting all this information together, one finds the reduced fusion coefficient:

$$\begin{aligned} \hat{C}^{+,-,+}_{n,m,q} &= \hat{C}^{-,-,-}_{n,m,q} \left[\frac{(d_1+1)(d_1+2)}{(q+1)(n+1)} \frac{(d_1-m+1)(d_1-m+2)}{nq}\right]^{1/2} \frac{d_1-m+1}{d_1} \\ &= \left(\frac{(d_1-m+1)^2}{mnq} \frac{(d_1+2)!\ (d_1-q)!(d_1-n)!(d_1-m+2)!}{(n+1)!\ (m-1)!\ (q+1)!}\right)^{1/2}, \end{aligned} \tag{C.6}$$

which agrees with (6.39).

Proceeding in a similar manner for other values of the indices $1_n, 1_m, 1_q$ we arrive at (6.39).

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